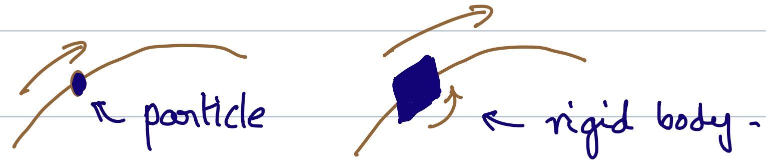
Course outline:

1. "Particles": objects whose orientation is not useful. and can be represented by a single point.

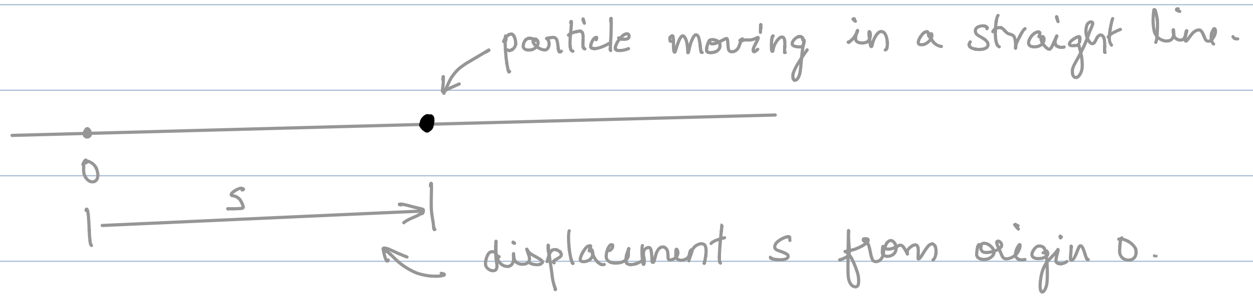
2. "Rigid bodies": objects whose position and orientation are important. Represented by positions and angles.

   "Rigid" $\hat{=}$ shape of object does not change.



3. "Deformable bodies": objects whose shape can change. Not studied in this course. See course on "Mechanics of Materials"

MOTION IN A STRAIGHT LINE



- Differentiation
  - position (s) $\longrightarrow$ velocity (v) $\longrightarrow$ acceleration (a).

    $$s \longrightarrow v = \frac{ds}{dt} \longrightarrow a = \frac{dv}{dt}.$$

- Integration to go backwards:
  - acceleration $\rightarrow$ velocity $\rightarrow$ position.

- **Integration: Acceleration to velocity**

$\boxed{\text{Case 1}}$: Say acceleration is known as a function of time t.

$$\frac{dv}{dt} = a(t)$$

$$\Rightarrow dv = a(t)\,dt \Rightarrow \int_{v_0}^{v} dv = \int_{t_0}^{t} a(t)\,dt.$$

where v_0, t_0 are initial velocity and time.

v is velocity at time t.

$$\int_{v_0}^{v} dv = [v]_{v_0}^{v} = v - v_0 = \int_{t_0}^{t} a(t)\,dt.$$

$$\boxed{v(t) = v_0 + \int_{t_0}^{t} a(t)\,dt}$$

Example

Say $a(t) = 3 + 4t$ m/s^2. given initial velocity $v_0 = 3\ m/s$ at initial time $t_0 = 0$ sec. Determine $v(t)$.

$$v(t) = v_0 + \int_{t_0}^{t} a(t)\,dt = 3 + \int_{0}^{t} (3+4t)\,dt$$

$$= 3 + \left[3t + \frac{4t^2}{2}\right]_0^t = 3 + 3t + 2t^2$$

$$\boxed{v(t) = 3 + 3t + 2t^2} \ m/s \quad \boxed{\text{ANS}}$$

Integration : velocity to position

Case 2 Say v is known as a function of time t.

$$v(t) = \frac{ds}{dt} \Rightarrow v(t)\,dt = ds$$

$$\Rightarrow \int_{s_0}^{s} ds = \int_{t_0}^{t} v(t)\,dt$$

$$\int_{s_0}^{s} ds = [s]_{s_0}^{s} = s(t) - s_0 = \int_{t_0}^{t} v(t)\,dt$$

$$\boxed{s(t) = s_0 + \int_{t_0}^{t} v(t)\,dt}$$

Integration : what if acceleration is known as a function of velocity v

Case 3 (rather than as a function of time)

why might acceleration be a funtion of v ?

eg. wind resistance, object moving through a fluid, electromagnetism (charges moving through a magnetic field) experience forces that depend on velocity.

So, $a = a(v) = \frac{dv}{dt}$

function of velocity

$$\Rightarrow \int_{v_0}^{v} dv = \underbrace{\int_{t_0}^{t} a(v)\,dt}_{} = v - v_0$$

True but useless because the integral depends on v, which may change with time, but we don't know <u>how</u> it changes with time.

- we need to isolate the variables! "separation of variables".

$$a(v) = \frac{dv}{dt}.$$

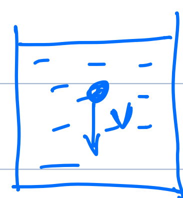
$$\int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{dv}{a(v)} \Rightarrow \boxed{\int_{v_0}^{v} \frac{dv}{a(v)} = t - t_0}$$

<u>Example</u>: Object falling through a liquid.

$t_0 = 0$ s, $v_0 = 0$ m/s. (initially at rest).

Given $a(v) = \underbrace{g}_{\text{gravity}} - \underbrace{cv}_{\text{linear drag}} = 10 - 3v$ say. | v in m/s. a in m/s^2.

<u>Determine $v(t)$</u>



$$a(v) = \frac{dv}{dt} \Rightarrow \int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{dv}{a(v)}$$

$$\Rightarrow t - t_0 = \int_{v_0}^{v} \frac{dv}{(10-3v)} = \left[\frac{\ln(10-3v)}{(-3)}\right]_{v_0}^{v}$$

Plug in $t_0 = 0$ and $V_0 = 0$.

$$t - 0 = \frac{\ln(10-3v)}{(-3)} - \frac{\ln(10)}{(-3)}.$$

$$-3t = \ln\left(\frac{10-3v}{10}\right).$$

Exponentiate both sides.

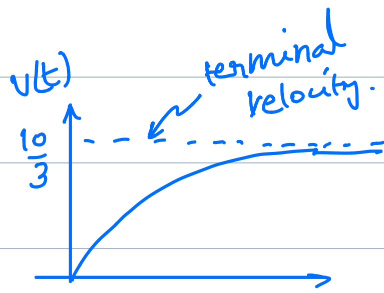
$$e^{-3t} = \frac{10-3v}{10} \quad \text{because } e^{\ln(\text{whatever})} = \text{whatever}.$$

$$10e^{-3t} = 10 - 3v \implies 3v = 10 - 10e^{-3t}$$

$$v = \frac{10 - 10e^{-3t}}{3}$$

$$\boxed{v(t) = \frac{10 - 10e^{-3t}}{3}}$$

← velocity as a function of time.



$\boxed{\text{ANS}}$

Integration: what if the acceleration is known only as a function of position?

$\boxed{\text{case 4}}$

Examples: gravity, electrostatics, spring, magnetism.

Other examples: feedback control in robots and many modern machines may have position dependent (or velocity dependent) forces from motors and other actuators.

eg. cruise control relies on applying forces that depend on the car's speed.

landing a rocket and keeping it vertical may require forces that depend on position, velocity, tilt angle, etc.

So, we have, $a(s) = \frac{dv}{dt}$. ← has 3 variables: v, s, t. need to convert to 2, so we can isolate them.

a as a function of s.

using chain rule,

$$a(s) = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v.$$

$$a(s) = \frac{dv}{ds} \cdot v. \Rightarrow a(s)\,ds = v\,dv$$

$$\int_{s_0}^{s} a(s)\,ds = \int_{v_0}^{v} v\,dv = \left[\frac{v^2}{2}\right]_{v_0}^{v} = \frac{v^2 - v_0^2}{2}.$$

$$\boxed{\int_{s_0}^{s} a(s)\,ds = \frac{v^2 - v_0^2}{2}}$$

Rearranging after integration gives us v as a function of s.

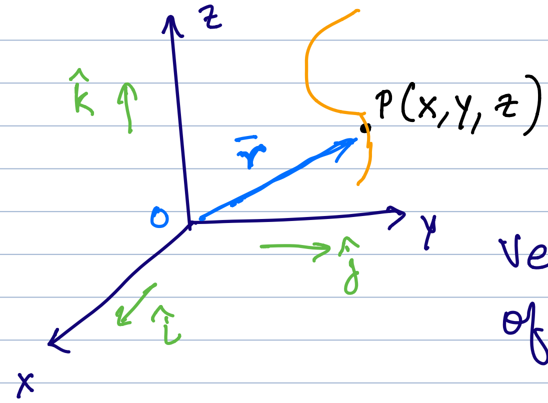
Integration: velocity as a function of positions

$$\frac{ds}{dt} = v(s) \Rightarrow \int_{s_0}^{s} \frac{ds}{v(s)} = \int_{t_0}^{t} dt = t - t_0. \Rightarrow \boxed{\int_{s_0}^{s} \frac{ds}{v(s)} = t - t_0}$$

separating variables

Motion in 2D and 3D: Cartesian coordinates.

Representation



$\hat{i}, \hat{j}, \hat{k}$ = unit vectors.

$\overline{OP} = \bar{r}$ = position vector.

$= x\hat{i} + y\hat{j} + z\hat{k}$.

Velocity of P $= \dfrac{d(\overline{OP})}{dt} = \dfrac{d\bar{r}}{dt} = \dfrac{d}{dt}\left(\quad \right) = \bar{v}$

$= \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$.

Recall chain rule of differentiation $\dfrac{d}{dt}(pq) = \dfrac{dp}{dt}q + p\dfrac{dq}{dt}$.

Note: $\dfrac{d\hat{i}}{dt} = \dfrac{d\hat{j}}{dt} = \dfrac{d\hat{k}}{dt} = 0$. (because constant vectors).

acceleration of P $= \dfrac{d\bar{v}}{dt} = \bar{a} = \dfrac{d^2\bar{r}}{dt^2} = \dfrac{d^2x}{dt^2}\hat{i} + \dfrac{d^2y}{dt^2}\hat{j} + \ldots$

$\ldots \dfrac{d^2z}{dt^2}\hat{k}$.

Notation alert: $\dfrac{dx}{dt} = \dot{x}$ ← dot represents one derivative.

$\ddot{x} = \dfrac{d^2x}{dt^2}$ ← 2 dots represents second derivative

Usually 'dots' represent derivatives with respect to time.

$$\bar{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k} \,. = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\bar{a} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k} \,. = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

x is subscipt

Again; $\bar{r} \to \bar{v} \to \bar{a}$ via differentiation.

$\bar{a} \to \bar{v} \to \bar{r}$ via integration.

We can sometimes do differentiation or integration one component at a time.

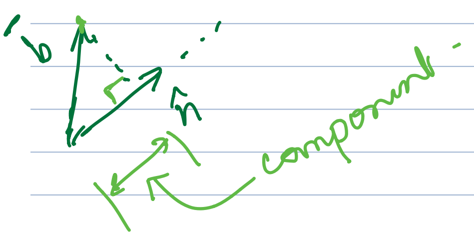
eg. $a_x \to v_x \to x$. by integration

$x \to v_x \to a_x$ by differentiation.

<u>Finding components of a vector along a direction given by unit vector $\hat{n}$.</u>

Given $\bar{b}$, some vector, it's component along $\hat{n}$

$$= \bar{b}\cdot\hat{n}.$$



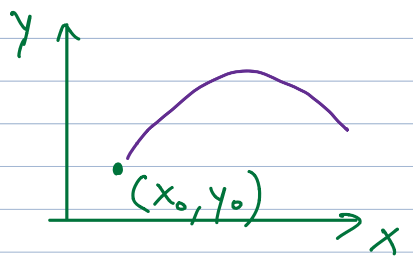
– $\underbrace{\text{Magnitude of vector}}$ $\bar{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$= |\bar{b}| = \|\bar{b}\| = \sqrt{b_x^2 + b_y^2 + b_z^2}\,.$$

(Pythagoras thm.)

$$\text{unit vector along } \bar{b} = \frac{\bar{b}}{|\bar{b}|} = \frac{b_x\hat{i} + b_y\hat{j} + b_z\hat{k}}{\sqrt{b_x^2 + b_y^2 + b_z^2}}$$

Example 1 : Projectile motion without drag. (2D).



Given acceleration :

along x direction: $a_x = \ddot{x} = 0$

along y direction: $a_y = \ddot{y} = -g$.

given initial position (x_0, y_0)

given initial velocity (V_{x0}, V_{y0})

determine $V_x(t)$, $V_y(t)$, $x(t)$, $y(t)$.

x - direction

$$a_x = 0 = \frac{dV_x}{dt}.$$

$a_x(t)$ to $V_x(t)$

$$\int_{V_{x0}}^{V_x} dV_x = \int_{t_0}^{t} a_x\,dt = \int_{t_0}^{t} 0\,dt = 0.$$

0 is like $a_x(t)$. can integrate wrt t.

$$[V_x]_{V_{x0}}^{V_x} = 0 \Rightarrow V_x - V_{x0} = 0$$

$\boxed{\text{ANS}}$ $\boxed{V_x(t) = V_{x0}}$ = constant velocity.

$V_x(t)$ to $x(t)$

$$\frac{dx}{dt} = V_x = V_{x0}.$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} V_{x0}\,dt = \left[V_{x0}\,t\right]_{t_0}^{t} = V_{x0}(t - t_0).$$

$$x - x_0 = v_{x_0}(t - t_0)$$

$$\boxed{x(t) = x_0 + v_{x_0}(t - t_0)} \quad \boxed{\text{ANS}}$$

y- direction

$a_y(t)$ to $v_y(t)$

$$a_y = \frac{dv_y}{dt} = -g. \Rightarrow dv_y = -g\,dt.$$

$$\int_{v_{y_0}}^{v_y} dv_y = -\int_{t_0}^{t} g\,dt = \left[-gt\right]_{t_0}^{t} = -g(t - t_0).$$

$$v_y - v_{y_0} = -g(t - t_0)$$

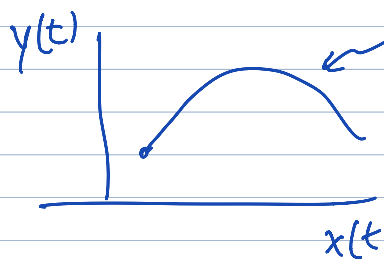
$$\boxed{v_y(t) = v_{y_0} - g(t - t_0)} \quad \boxed{\text{ANS}}$$

$$x(t) = x_0 + \int_{t_0}^{t} v_y(t)\,dt = x_0 + \int_{t_0}^{t} \left[v_{y_0} - g(t - t_0)\right] dt$$

$$= x_0 + \left[v_{y_0} t\right]_{t_0}^{t} - \left[g\,\frac{1}{2}(t - t_0)^2\right]_{t_0}^{t}$$

$$\boxed{x(t) = x_0 + v_{y_0}(t - t_0) - \frac{g}{2}(t - t_0)^2} \quad \boxed{\text{ANS}}$$

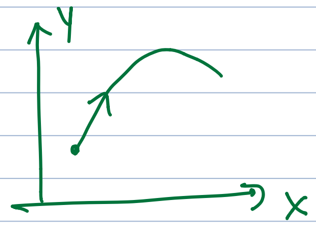
Plot $x(t)$ vs $y(t)$



motion is a parabola.

Example 2 : Projectile with linear drag

A ball is thrown up at some angle in the x-y plane. (y is vertical). The acceleration is given by the equation : $\bar{a} = -g\hat{j} - c\bar{v}$. — (1)



Given (x_0, y_0) and (v_{x0}, v_{y0}), determine $v_x(t)$, $v_y(t)$, $x(t)$, $y(t)$. at $t_0 = 0$.

We are given a vector equation for $\bar{a} = -g\hat{j} - c\bar{v}$.

Let us get scalar equations for the components a_x and a_y.

Noting that $\bar{a} = a_x\hat{i} + a_y\hat{j}$ and $\bar{v} = v_x\hat{i} + v_y\hat{j}$,

Plug these in (1)

$$a_x\hat{i} + a_y\hat{j} = -g\hat{j} - c(v_x\hat{i} + v_y\hat{j})$$

$$a_x\hat{i} + a_y\hat{j} = -cv_x\hat{i} - g\hat{j} - cv_y\hat{j}$$

$$(a_x)\hat{i} + (a_y)\hat{j} = (-cv_x)\hat{i} + (-g - cv_y)\hat{j}$$

Equating $\hat{i}$ terms :

$$\boxed{a_x = -cv_x} \quad (2)$$

Equating $\hat{j}$ terms

$$\boxed{a_y = -g - cv_y} \quad (3)$$

Note: the accelerations are known as functions of velocity

x-direction.

$$a_x = -c v_x$$

$$\frac{dv_x}{dt} = -c v_x$$

$$\int_{v_{x0}}^{v_x} \frac{dv_x}{v_x} = \int_{t_0}^{t} -c\,dt$$

$$\Big[\ln(v_x)\Big]_{v_{x0}}^{v_x} = -c\,[t]_{t_0}^{t}$$

$$\ln(v_x) - \ln(v_{x0}) = -c\,[t - t_0]. \quad \text{Say } t_0 = 0 \text{ (given).}$$

$$\ln\left(\frac{v_x}{v_{x0}}\right) = -ct.$$

Exponentiate both sides, $\left(e^{\ln(p)} = p\right)$.

$$e^{\ln(v_x/v_{x0})} = \frac{v_x}{v_{x0}} = e^{-ct}$$

$$v_x = v_{x0}\, e^{-ct} \longleftarrow \text{ANS.}$$

$$\boxed{v_x(t) = v_{x0}\, e^{-ct}} \quad \boxed{\text{ANS}}$$

To get $x(t)$,

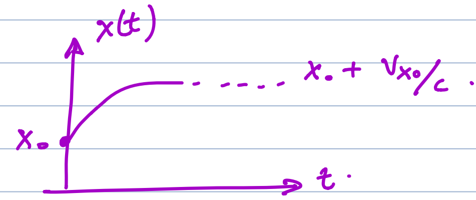
$$x(t) = x_0 + \int_0^t v_x(t)\,dt$$

$$= x_0 + \int_0^t v_{x0}\, e^{-ct}\,dt$$

$$= x_0 + v_{x0}\int_0^t e^{-ct}\,dt = x_0 + v_{x0}\left[\frac{e^{-ct}}{-c}\right]_0^t$$

$$= x_0 + v_{x0}\left[\frac{e^{-ct}}{-c} - \frac{e^{0}}{-c}\right] = x_0 + v_{x0}\left[\frac{e^{-ct}-1}{-c}\right]$$

$$\boxed{x(t) = x_0 - \frac{v_{x0}}{c}\left(e^{-ct}-1\right)}$$



y- direction.

$$a_y = -g - cv_y = \frac{dv_y}{dt}.$$

$$\int_{v_{y0}}^{v_y} \frac{dv_y}{-g-cv_y} = \int_{t_0 \approx 0}^{t} dt$$

$$\left[\frac{\ln(-g-cv_y)}{-c}\right]_{v_{y0}}^{v_y} = t - 0 = -\frac{1}{c}\ln\left[\frac{-g-cv_y}{-g-cv_{y0}}\right]$$

$$\ln\left[\frac{g+cv_y}{g+cv_{y0}}\right] = -ct \Rightarrow \frac{g+cv_y}{g+cv_{y0}} = e^{-ct}$$

$$g + cv_y = (g+cv_{y0})e^{-ct}.$$

$$\boxed{v_y(t) = \frac{-g + (g+cv_{y0})e^{-ct}}{c}}$$ $\boxed{\text{ANS}}$

This is a function of time. We can integrate directly to get $y(t)$.

$$y(t) = y_0 + \int_0^t v_y(t)\,dt$$

$$= y_0 + \int_0^t \left[\frac{-g + (g + cv_{y_0})e^{-ct}}{c} \right] dt$$

$$= y_0 - gt/c - \frac{g + cv_{y_0}}{c} \cdot \left[e^{-ct} \right]_0^t$$

$$\boxed{y(t) = y_0 - gt/c - \left(\frac{g + cv_{y_0}}{c} \right) \left(e^{-ct} - 1 \right).}$$

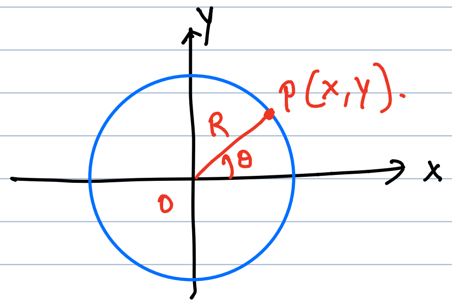
ANS

Other coordinate systems in 2D

(1) Tangential - Normal coordinates

(2) Polar coordinates

Special case of (1) and (2) : Motion in a circle



R - radius . θ - angular position.

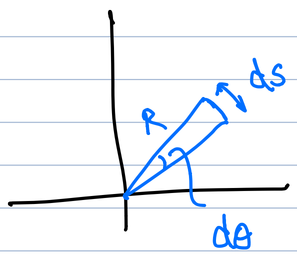
$x = R\cos\theta$

$y = R\sin\theta$.

Say the particle moves with angular rate $\dot{\theta} = \omega$.

Speed v of the particle, $v = R\dot{\theta} = R\omega$

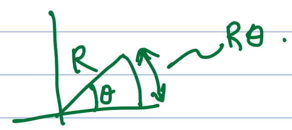
The velocity is tangential to the circle.



Distance traveled in time dt

$ds = R\,d\theta$,

where $d\theta$ is angle change in time dt .

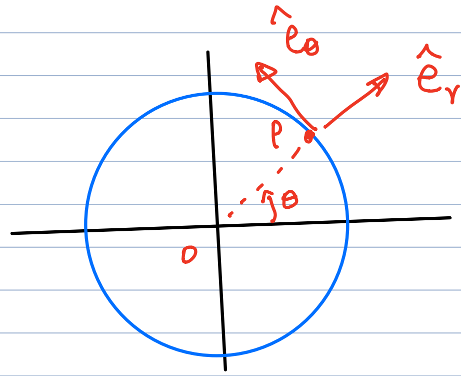


full circle, $\theta = 2\pi$.
circumference $= 2\pi R$.

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{ds}{dt} = R\frac{d\theta}{dt}$$

(You can instead use Δt, $\Delta\theta$, Δs, $v = \lim_{\Delta t \to 0} R\frac{\Delta\theta}{\Delta t} = R\frac{d\theta}{dt}$.) .

$$\boxed{v = R\dot{\theta}}$$



unit vectors.

POLAR COORDINATES

$\hat e_r$ - radial unit vector

$\hat e_\theta$ - azimuthal unit vector.

special case.

position vector $\bar r = \overline{OP} = x\hat i + y\hat j = R\hat e_r$ $\quad \boxed{\bar r = R\hat e_r}$ (1)

velocity $\boxed{\bar v = R\dot\theta\,\hat e_\theta \,.}$ (2)

Note: $\bar v = \frac{d\bar r}{dt} = \frac{d}{dt}\left(R\hat e_r\right) = R\frac{d\hat e_r}{dt} = R\dot\theta\,\hat e_\theta$ from (2)

$$\cancel{R}\frac{d\hat e_r}{dt} = \cancel{R}\ddot\theta\frac{d\hat e_\theta}{dt} \Rightarrow \boxed{\frac{d\hat e_r}{dt} = \dot\theta\,\hat e_\theta}$$

check $\boxed{\frac{d\hat e_\theta}{dt} = -\dot\theta\,\hat e_r \,.}$

acceleration $\bar a$

$$\bar a = \frac{d\bar v}{dt} = \frac{d}{dt}\left(R\dot\theta\,\hat e_\theta\right) = R\frac{d}{dt}\left(\dot\theta\,\hat e_\theta\right)$$

$$= R\ddot\theta\,\hat e_\theta + R\dot\theta\frac{d}{dt}(\hat e_\theta)$$

product rule/ chain rule.

$$= R\ddot\theta\,\hat e_\theta + R\dot\theta\left(-\dot\theta\,\hat e_r\right)$$

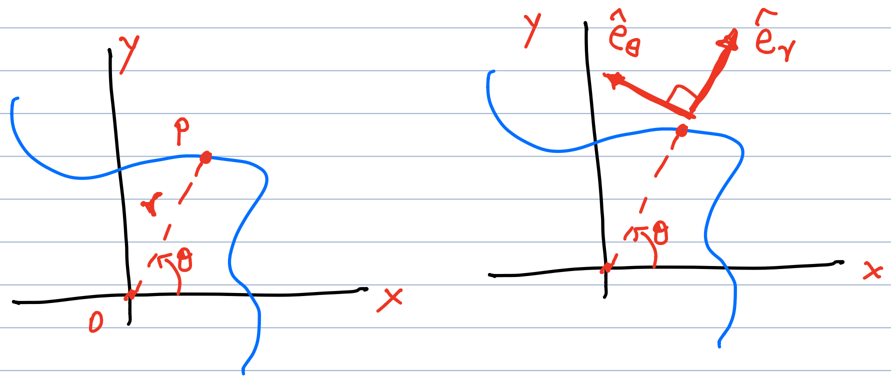
$$\bar a = -R\dot\theta^2\,\hat e_r + R\ddot\theta\,\hat e_\theta$$

$$\boxed{\bar{a} = \underbrace{-R\dot{\theta}^2\,\hat{e}_r}_{\text{centripetal acceleration}} + \underbrace{R\ddot{\theta}\,\hat{e}_\theta}_{\text{tangential or azimuthal acceleration}}}$$

acceleration due to direction of $\bar{v}$ changing

acceleration due to speed v (magnitude of $\bar{v}$) changing.

Polar coordinates for general motion in 2D



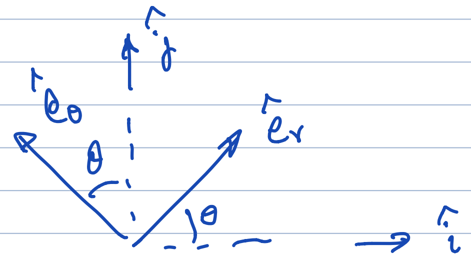
position vector $\boxed{\bar{r} = r\,\hat{e}_r}$

velocity $\boxed{\bar{v} = \underbrace{\dot{r}\,\hat{e}_r}_{\text{velocity along } \hat{e}_r} + \underbrace{r\dot{\theta}\,\hat{e}_\theta}_{\text{along } \hat{e}_\theta}}$

acceleration $\boxed{\bar{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta}$

- centripetal acceleration $= -r\dot{\theta}^2\hat{e}_r$

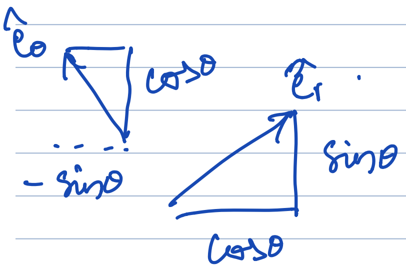
- coriolis acceleration $= 2\dot{r}\dot{\theta}\,\hat{e}_\theta$.



Converting polar $\leftrightarrow$ cartesian

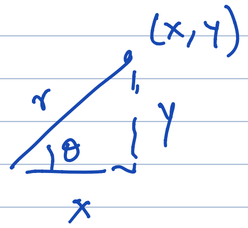
$$\hat{e}_r = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j}$$

$$\hat{e}_\theta = (-\sin\theta)\,\hat{i} + (\cos\theta)\,\hat{j}$$



$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta.$$



$\Rightarrow$ $x = r\cos\theta$

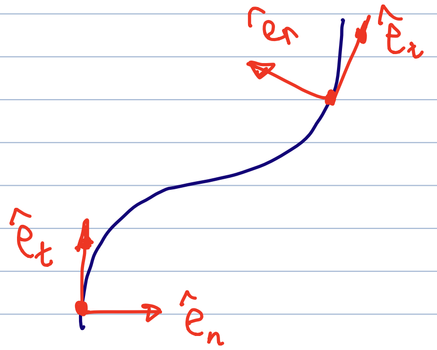
$y = r\sin\theta$.

$$\sqrt{x^2+y^2} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2} = r$$

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= y/x \end{aligned}}$$

Recall: you can use the signs of $\sin\theta = x/r$ $\cos\theta = y/r$ to determine the quadrant of θ.

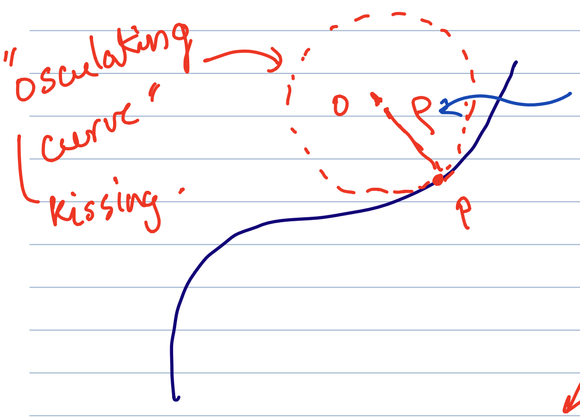
Tangential - normal coordinates : 2D general motion.



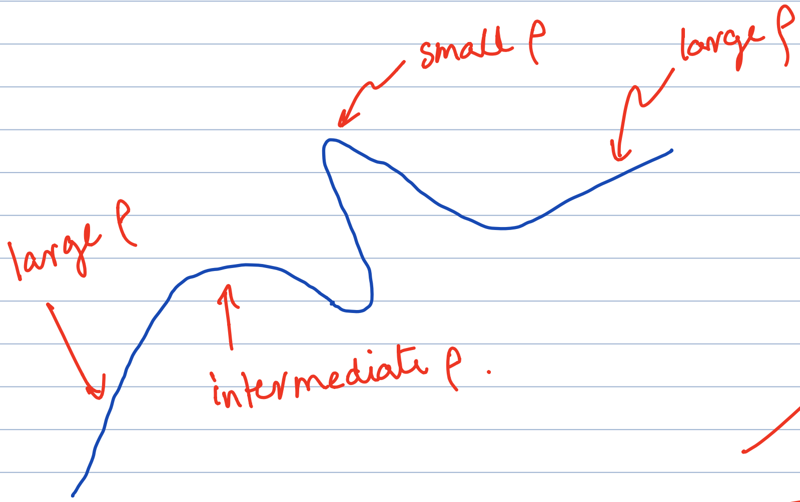
$\hat{e}_t$ - tangential unit vector

$\hat{e}_n$ - normal unit vector. always points toward the center of curvature and $\perp^{ar}$ to $\hat{e}_t$.

$\hat{e}_n$ - is not unique at an inflection point (when curvature changes sign).

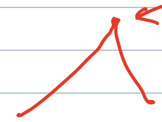


ρ : Greek letter "rho". radius of curvature (instantaneous " " ").



st. line

$\rho = ?\ \infty$



point

$\rho = ?\ 0$

velocity

$$\boxed{\bar{v} = \underbrace{v}_{\substack{\text{magnitude} \\ (v>0)}}\ \underbrace{\hat{e}_t}_{\substack{\text{unit vector} \\ \text{direction}}}}$$

acceleration

$$\boxed{\bar{a} = \underbrace{\dot{v}\,\hat{e}_t}_{\substack{\text{term due} \\ \text{to speed} \\ \text{or magnitude change}}} + \underbrace{\overbrace{\frac{v^2}{\rho}\,\hat{e}_n}^{\text{centripetal.}}}_{\substack{\text{direction} \\ \text{change}}}}$$

Helpful rearrangements :

$$\hat{e}_t = \frac{\bar{v}}{v} = \frac{v_x \hat{i} + v_y \hat{j}}{\sqrt{v_x^2 + v_y^2}} .$$

$$|\bar{a}| = \text{magnitude of } \bar{a} = \sqrt{(\dot{v})^2 + \left(\frac{v^2}{\rho}\right)^2}$$

Ex 1 particle is moving along a curved path.
- magnitude of acceleration $|\bar{a}| = 20 \ m/s^2$.
- rate of change of speed $= 5 \ m/s^2$.
- current speed $= 2 \ m/s$.

Determine the current radius of curvature. ρ.

Tangential - normal coordinates:

$$\bar{a} = \dot{v} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n . \quad \text{- acceleration vector.}$$

$$|\bar{a}| = \sqrt{(\dot{v})^2 + \left(\frac{v^2}{\rho}\right)^2} = 20 \ m/s^2$$

$\dot{v} = 5 \ m/s^2$, $v = 2 \ m/s$.

$$\sqrt{5^2 + \left(\frac{2^2}{\rho}\right)^2} = 20 \Rightarrow 5^2 + \frac{4^2}{\rho^2} = 400$$

$$4^2/\rho^2 = 400 - 25 = 375.$$

$$\rho = \sqrt{4^2/375} = \_\_\_$$ compute decimal representation for HW and exams.

Example 2: Given $x(t) = 5\sin(3t)$, $y(t) = 3t$, t in s, x, y in m.

determine:

(1) $\bar{v}$ in cartesian.

(2) magnitude of $\bar{v}$.

(3) $\bar{a}$ in cartesian

(4) $\bar{a}$ in tangential - normal

(5) $\dot{v}$

(6) radius of curvature ρ.

$t = 1\,s$.

(1) $\bar{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$, $\dot{x} = 5 \cdot 3 \cdot \cos(3t) = 15\cos 3t$

$\dot{y} = 3.$

$$\boxed{\bar{v} = 15\cos 3t\,\hat{i} + 3\,\hat{j}} \quad (1)$$

ANS $-14.85\,\hat{i} + 3\,\hat{j}$ (m/s) $=$ $\boxed{\bar{v} = 15\cos(3)\,\hat{i} + 3\,\hat{j}}$ $\boxed{\text{ANS (1)}}$

(2) $|\bar{v}|$ magnitude $= \sqrt{(15\cos(3))^2 + 3^2} = \_\_\_$ ANS (2).

$= 15.15$ m/s.

(3) $\bar{a}$ in cartesian $= -45\sin 3t\,\hat{i} + 0\,\hat{j}$.

$\bar{a}(t=1s) = -45\sin(3)\,\hat{i} = -6.35\,\hat{i}$ m/s^2 ANS (3).

(4) $\bar{a}$ in tangential - normal:

$$\bar{a} = -45 \sin(3)\,\hat{i} = -6.35\,\hat{i}$$

replace in terms of $\hat{e}_t$ and $\hat{e}_n$.

$$\hat{e}_t = \frac{\bar{v}}{|\bar{v}|} = \frac{15\cos 3\,\hat{i} + 3\hat{j}}{\sqrt{(15\cos 3)^2 + 3^2}}$$

$$= \frac{-14.85\,\hat{i} + 3\hat{j}}{15.15} = -0.98\,\hat{i} + 0.198\,\hat{j}$$

Tangential acceleration (component). $a_t = \bar{a} \cdot \hat{e}_t$

$$= (-6.35\,\hat{i}) \cdot (-0.98\,\hat{i} + 0.198\,\hat{j})$$

dot product.

$$= (6.35)(0.98)(\hat{i} \cdot \hat{i}) + \bar{0}$$

$$= 6.223 \text{ m/s}^2.$$

Total acceleration $= \bar{a} = a_t \hat{e}_t + a_n \hat{e}_n$

$$-6.35\,\hat{i} = 6.223 \cdot (-0.98\,\hat{i} + 0.198\,\hat{j}) + a_n \hat{e}_n$$

$$a_n \hat{e}_n = -6.35\,\hat{i} + (0.98) \cdot (6.223)\,\hat{i} - (0.198)(6.223)\,\hat{j}$$

$$a_n \hat{e}_n = -0.25\,\hat{i} - 1.23\,\hat{j}$$

$$a_n = \sqrt{0.25^2 + 1.23^2} = 1.26 \text{ m/s}^2$$

$$\hat{e}_n = -0.2\,\hat{i} - 0.98\,\hat{j}.$$

$$\boxed{\bar{a} = 6.223\,\hat{e}_t + 1.26\,\hat{e}_n}$$

$$\bar{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n.$$

$\Rightarrow$ (5) $\dot{v} = 6.223\ m/s^2$

(6) $\rho = ?$ Use $|\bar{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2}$

(or) use $a_n = 1.26 = v^2/\rho. = \frac{15.15^2}{\rho}.$

$$\rho = \frac{15.15^2}{1.26} = \underline{\underline{182.16\ m}}$$

If you have a vector e_t = a i + b j
then a perpendicular to this vector is -b i + a j
because the dot product is a(-b)+a*b = 0.

b i - a j is also a perpendicular.
So e_n = perpendicular e_t and is one of the
above two perpendicular vectors to e_t.

RELATIVE MOTION

There is no such thing as absolute motion.
Implicitly, when we report $\bar{v}$ or $\bar{a}$, we are often assuming that those $\bar{v}$ or $\bar{a}$ are relative to the surface of the earth. ← assume if not specified.

e.g. car at 65 mph. ⇒ relative to road surface.

We can ONLY speak of relative motion.

key principles:

Two objects A and B.

Define $\boxed{\bar{v}_{A/B} = \underbrace{\bar{v}_A - \bar{v}_B}}$ ← velocity of A relative to B.

relative to a common reference. eg. earth surface.

Vector equations

and does not work for velocity magnitude in general.

Three objects A, B, C.

given $\bar{v}_{A/C}$ and $\bar{v}_{B/C}$,

then relative velocity $\boxed{\bar{v}_{A/B} = \bar{v}_{A/C} - \bar{v}_{B/C}}$

$$\boxed{\bar{v}_{B/A} = -\bar{v}_{A/B}} = \bar{v}_B - \bar{v}_A = \bar{v}_{B/C} - \bar{v}_{A/C}.$$

these equations are called Galilean relativity. (Galileo)
as opposed to Einstein's theory of relativity.

Acceleration

$$\bar{a}_{A/B} = \underbrace{\bar{a}_A - \bar{a}_B}$$

relative to earth's surface

$$\bar{a}_{A/B} = \bar{a}_{A/C} - \bar{a}_{B/C}.$$

$$\bar{a}_{B/A} = -\bar{a}_{A/B}.$$

$\bar{a}_{A/B}$ read as "acceleration of A with respect to B."
" " " relative to "

Examples: 1) Given $\bar{V}_A = 2\hat{i}$ (m/s). } with respect to earth's surface
$\bar{V}_B = -3\hat{j}$ (m/s).

then compute $\bar{V}_{A/B}$, $\bar{V}_{B/A}$.

$$\bar{V}_{A/B} = \bar{V}_A - \bar{V}_B = 2\hat{i} - (-3\hat{j}) = 2\hat{i} + 3\hat{j}.$$

$$\bar{V}_{B/A} = \bar{V}_B - \bar{V}_A = -\bar{V}_{A/B} = -2\hat{i} - 3\hat{j}.$$

Example 2: given $\bar{V}_A = 2\hat{i}$, $V_B = -3\hat{i}$ (m/s)

what is $\bar{V}_{A/B}$?

$$\bar{V}_{A/B} = 2\hat{i} - (-3\hat{i}) = 5\hat{i} \text{ (m/s)}.$$

Example 3.

Given $\bar{V}_{A/B} = 3\hat{i} + 4\hat{j}$. $\bar{V}_{A/C} = 2\hat{i} - 3\hat{j}$)

compute $\bar{V}_{B/C}$, $\bar{V}_{C/A}$.

$$\bar{V}_{C/A} = -\bar{V}_{A/C} = -(2\hat{i} - 3\hat{j}) = \underline{-2\hat{i} + 3\hat{j}} \ (m/s).$$

ANS

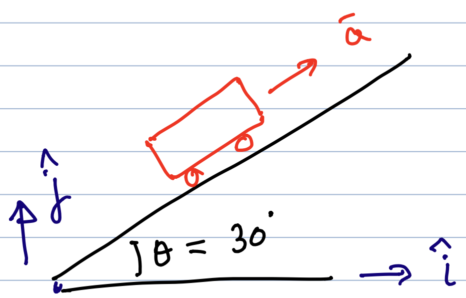
$$\bar{V}_{B/C} = \bar{V}_{B/A} - \bar{V}_{C/A}$$

$$= -\bar{V}_{A/B} - (-2\hat{i} + 3\hat{j}). = -(3\hat{i} + 4\hat{j}) - (-2\hat{i} + 3\hat{j})$$

$$= -3\hat{i} - 4\hat{j} + 2\hat{i} - 3\hat{j} = \underline{-\hat{i} - 7\hat{j}} . \ (m/s).$$

ANS

Example 4



Car on a slope ($\theta = 30°$) has acceleration $\bar{a}$ with magnitude $5\ m/s^2$. with respect to the earth.

Someone drops a ball within the car. and this ball has acceleration $g = 9.81\ m/s^2$ in the vertical direction (downward) with respect to the earth.

Determine $\bar{a}_{car/earth}$, $\bar{a}_{ball/car}$. - vectors.

Given: $\bar{a}_{ball/earth} = -9.81\,\hat{j} \ \ m/s^2$ (1)

$$|\bar{a}_{car/earth}| = 5\ m/s^2 \quad (2)$$

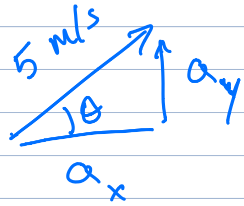
(3) $\bar{a}_{car/earth} = a_x \hat{i} + a_y \hat{j}$. unknown components.

Using (3) and (2), $\sqrt{a_x^2 + a_y^2} = 5$

$$a_x^2 + a_y^2 = 5^2 = 25.$$

$\bar{a}_{car/earth}$ is along slope.

$\Rightarrow$



$$\left.\begin{aligned} a_x &= 5\cos\theta \\ a_y &= 5\sin\theta \end{aligned}\right] \ (4)$$

Use (4) in (3)

$$\bar{a}_{car/earth} = 5\cos\theta\,\hat{i} + 5\sin\theta\,\hat{j} . \quad (5)$$

$$= 5\cos 30^\circ\,\hat{i} + 5\sin 30^\circ\,\hat{j} \quad (m/s^2)$$

$$\underline{\bar{a}_{car/earth} = 4.33\,\hat{i} + 2.5\,\hat{j}} \quad \text{ANS}$$

$$\bar{a}_{ball/car} = \bar{a}_{ball/earth} - \bar{a}_{car/earth}$$

$$= -9.81\,\hat{j} - (4.33\,\hat{i} + 2.5\,\hat{j})$$

$$\underline{\bar{a}_{ball/car} = -4.33\,\hat{i} - 12.31\,\hat{j}} \quad m/s^2$$

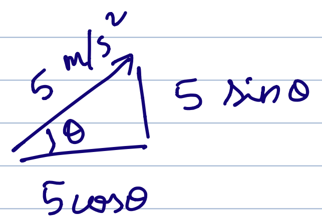
ANS

Example 5. Above problem rearranged. (some of the HW problems).

- car accelerates up a slope θ. θ not known.
- given $|\bar{a}_{car/earth}| = 5 \ m/s^2$
- given $\bar{a}_{ball/earth} = -9.81 \ m/s^2$
- given $|\bar{a}_{ball/car}| = 13.05 \ m/s^2$.

Determine acceleration of $\bar{a}_{ball/car}$, $\bar{a}_{car/earth}$ etc.

$$\bar{a}_{car/earth} = 5\cos\theta\,\hat{i} + 5\sin\theta\,\hat{j}. \quad (1)$$



$$\bar{a}_{ball/car} = \bar{a}_{ball/earth} - \bar{a}_{car/earth}$$

$$= -9.81\hat{j} - (5\cos\theta\,\hat{i} + 5\sin\theta\,\hat{j})$$

$$(2) \quad = -5\cos\theta\,\hat{i} - (9.81 + 5\sin\theta)\hat{j}.$$

$$|\bar{a}_{ball/car}| = 13.05 = \sqrt{(5\cos\theta)^2 + (9.81 + 5\sin\theta)^2}$$

- solve for θ. -

$$13.05^2 = 5^2\cos^2\theta + 9.81^2 + 5^2\sin^2\theta + 2.(9.81).5\sin\theta$$

$$= 5^2 + 9.81^2 + 2(9.81).5.\sin\theta.$$

$$\sin\theta = (13.05^2 - 5^2 - 9.81^2)/((2)(9.81)(5))$$

$\theta = \sin^{-1}\left(- \right)$ ANS (3)

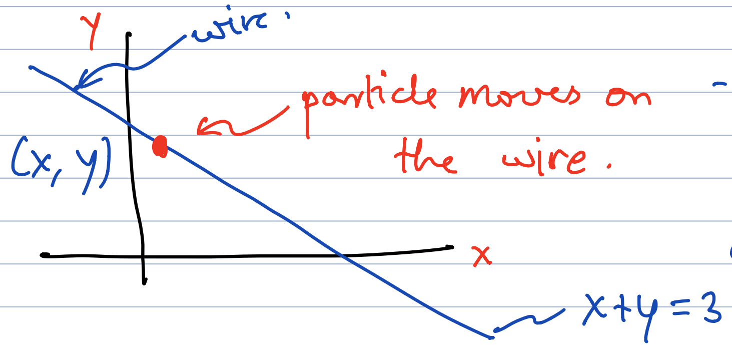
Plug θ in (1) $\bar{a}_{car/earth}$.

Plug θ in (2) to get $\bar{a}_{ball/car}$.

ANS.

CONSTRAINED MOTION.

CONSTRAINT = RESTRICTION ON MOTION



- particle moves on line $x + y = 3$. (1)

a) determine the relation between v_x and v_y of particle.

b) determine relation between a_x and a_y of particle.

$$v_x = \dot{x}, \quad v_y = \dot{y}, \quad a_x = \ddot{x}, \quad a_y = \ddot{y}$$

- Solution : $x + y = 3$.

a) Differentiate with respect to time.

$$\frac{d}{dt}(x+y) = \frac{d}{dt}(3)$$

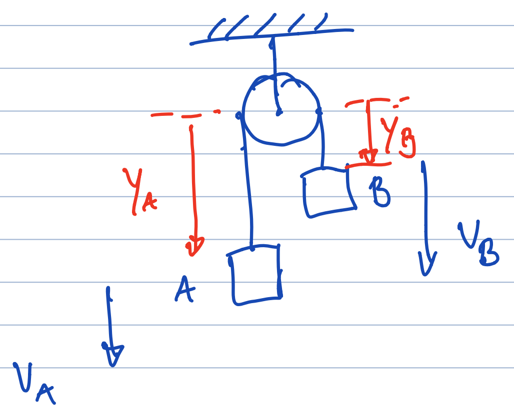
ANS $\boxed{\dot{x} + \dot{y} = 0}$ (or) $\boxed{v_x + v_y = 0}$ $\boxed{v_x = -v_y}$

(b) Diff once more

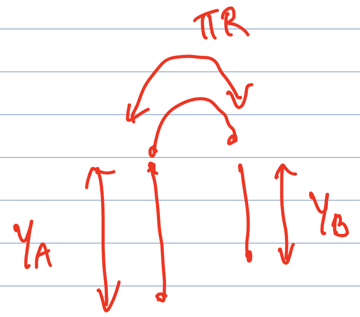
$$\boxed{\ddot{x} + \ddot{y} = 0} \Rightarrow \boxed{a_x + a_y = 0}$$ ANS b.

Ex. Simple pulley.

Assume the rope is inextensible,



- what is the relationship between V_A and V_B?
- equivalently, what is the relation between $\dot{y}_A$ and $\dot{y}_B$.



Total length of string $= y_A + \pi R + y_B$

$= \text{constant}$

$$y_A + \pi R + y_B = \text{const}$$

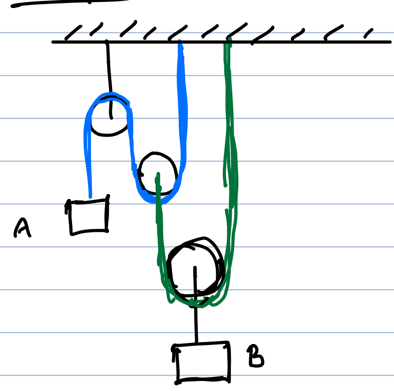
$$\boxed{y_A + y_B = \text{const}}$$

$$\dot{y}_A + \dot{y}_B = 0 \Rightarrow V_A + V_B = 0$$

$$\boxed{V_A = -V_B}$$

$$\boxed{a_A = -a_B}$$

Example 2 :



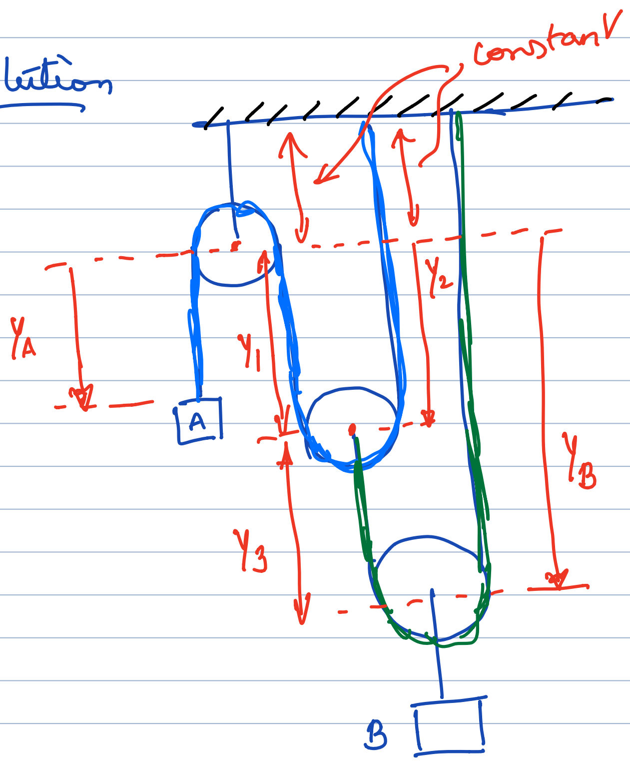
- Find the velocity of B given velocity of A is 2m/s upward.
- Find acceleration of A given acceleration of B is 3 m/s² downward.

"constrained motion"

All ropes are inextensible ≡ length is constant.

To do: find relation between v_A and v_B. &
" " " a_A and a_B.

Solution



Step 1: Define variables for positions of key masses. Here A and B.

Step 2: Define variables for other segments of rope as necessary.

Step 3: Write down expressions for lengths of various ropes.

Blue rope: $y_A + \pi R_1 + y_1 + \pi R_2 + y_2 + \text{const} = \text{total length}$
$= \text{const}.$

Differentiate:

$$\dot{y}_A + 0 + \dot{y}_1 + 0 + \dot{y}_2 + 0 = 0$$

$$\dot{y}_A + \dot{y}_1 + \dot{y}_2 = 0 \quad (1)$$

Because $y_1 = y_2 \Rightarrow \dot{y}_1 = \dot{y}_2$ (2)

$$\dot{y}_A + 2\dot{y}_1 = 0 \quad (3)$$

Green rope: $y_3 + \pi R_3 + y_B + \text{const.} = \text{total length}$
$= \text{constant.}$

Differentiate:

$$\dot{y}_3 + 0 + \dot{y}_B + 0 = 0.$$

$$\dot{y}_B = -\dot{y}_3 \quad (4)$$

$$y_1 + y_3 = y_B$$

$$\Rightarrow \dot{y}_1 + \dot{y}_3 = \dot{y}_B \quad \Rightarrow \quad \dot{y}_3 = \dot{y}_B - \dot{y}_1 \quad (5)$$

Plug (5) in (4)

$$\dot{y}_B = -(\dot{y}_B - \dot{y}_1) = -\dot{y}_B + \dot{y}_1$$

$$\Rightarrow 2\dot{y}_B = \dot{y}_1 \quad (6)$$

Recall (3) $\dot{y}_A + 2\dot{y}_1 = 0$

(6) in (3) $\Rightarrow \dot{y}_A + 2(2\dot{y}_B) = 0$

$$(7) \quad \boxed{\dot{y}_A = -4\dot{y}_B}$$

This is a relation between v_A and v_B.

where $v_A = \dot{y}_A$, $v_B = \dot{y}_B$.

Because A moves up & y_A defined downward.

$$\dot{y}_A = -2\,m/s. \Rightarrow \dot{y}_B = -\dot{y}_A/(-4) = \frac{-2}{-4} = 0.5\,m/s$$

$$\boxed{\dot{y}_B = 0.5 \text{ m/s}}$$ ANS

(B moving downward because y_B defined downward positive).

For acceleration: differentiate (7)

$$\boxed{\ddot{y}_A = -4\ddot{y}_B}$$

given B accelerates downward at 3 m/s^2

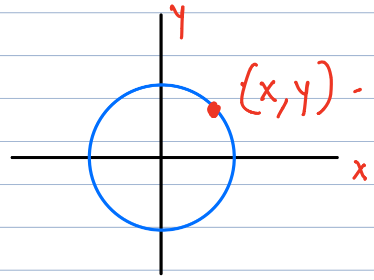
$\Rightarrow \ddot{y}_B = 3 \text{ m/s}^2$.

$$\ddot{y}_A = -4(3) = \boxed{-12 \text{ m/s}^2.}$$ ANS

A accelerating upward at 12 m/s^2.

Example: A particle moves in a circle

$$x^2 + y^2 = 5. \quad (1)$$



(a) Determine a relation between $x, y, \dot{x}, \dot{y}$.

(b) Determine a relation between $x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}$.

(a) Differentiate (1) with respect to time t.

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(5) = 0. \quad (2)$$

Recall chain rule: $\frac{d}{dt}(f(x)) = \frac{df}{dx} \cdot \frac{dx}{dt}$.

$$\frac{d}{dt}\left(g(y)\right) = \frac{dg}{dy} \cdot \frac{dy}{dt}.$$

So, $\dfrac{d(x^2)}{dt} = \dfrac{d(x^2)}{dx} \cdot \dfrac{dx}{dt} = 2x \cdot \dot{x}$ (3)

Similarly. $\dfrac{d}{dt}(y^2) = \dfrac{d}{dy}(y^2) \cdot \dfrac{dy}{dt} = 2y \cdot \dot{y}.$ (4)

Use (3) & (4) in (2)

$$\frac{d}{dt}(x^2+y^2) = 0$$

$$\cancel{2}x\dot{x} + \cancel{2}y\dot{y} = 0$$

(5) $\boxed{x\dot{x} + y\dot{y} = 0}$. ANS (a)

Given 3, can find the 4^{th} variable.

(b) find relation between $x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}$.

Recall multivariate chain rule:

$$\frac{d}{dt}\left(h(p,q)\right) = \frac{\partial h}{\partial p} \cdot \frac{dp}{dt} + \frac{\partial h}{\partial q} \cdot \frac{dq}{dt}.$$

$\dfrac{d}{dt}(x\dot{x}) = h(p,q)$ where $p = x$. $q = \dot{x}$.

$= pq.$

$$= \frac{\partial(pq)}{\partial p}\frac{dp}{dt} + \frac{\partial(pq)}{\partial q} \cdot \frac{dq}{dt}$$

$$= q \cdot \dot{p} + p\dot{q}$$

$$\frac{d}{dt}(x\dot{x}) = \dot{x}\cdot\dot{x} + x\ddot{x}$$

$$= \dot{x}^2 + x\ddot{x}$$

Similarly: $\frac{d}{dt}(y\dot{y}) = \dot{y}^2 + y\ddot{y}$

differentiating 5

$$\frac{d}{dt}\left(x\dot{x} + y\dot{y}\right) = 0$$

$$\boxed{\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} = 0}$$

ANS b.

Thus far, what we have covered : "KINEMATICS"

cartesian, polar, tangential-normal, relative motion, constrained motion.

kinematics ≡ " calculus and geometry applied to motion". without considering forces

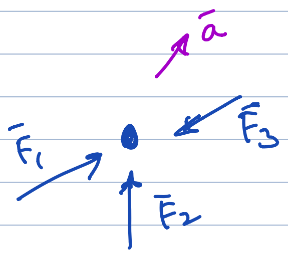
TODAY : "KINETICS" : relation between forces and motion.

↳ "$F = ma$".

NEWTON'S SECOND LAW:

applicable for a single body or a collection of bodies.

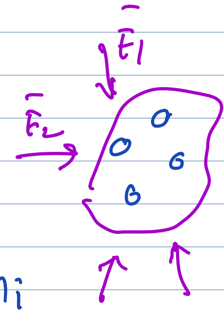
For any body, say, a particle, the sum of all forces on this particle equals mass times acceleration.



$$\sum \bar{F}_i = (\text{mass } m) \cdot \times \left(\begin{array}{c}\text{acceleration } \bar{a} \\ \text{of the particle}\end{array}\right)$$

sum of all forces

For any collection of objects with masses $m_1, m_2 \ldots$ the sum of all external forces on these objects



External $\sum \bar{F} = m_1 \bar{a}_1 + m_2 \bar{a}_2 + m_3 \bar{a}_3 + \ldots$

where $\bar{a}_i$ acceleration of the mass m_i

Procedure for solving particle dynamics problems

(1) Draw FBDs. (Free Body Diagrams).

- As many as necessary. Usually, one for each mass/particle in the problem

(2) Define coordinate systems / sign conventions

(3) For each FBD or mass or particle,

Calculate sum of all forces

(4) For each FBD/mass/particle, write an expression for the acceleration: cartesian, polar, tangential-normal.

(5) For each FBD, ... do $\sum \bar{F} = m\bar{a}$.

—

Few quantities given, few unknowns.

↙

write as many equations as unknowns.

– SOLVE –

→ Sometimes need to use "constrained motion" to get additional equations relating different accelerations. eg. for pulleys.

A couple of things:

(1) How accurate is "$\sum \bar{F} = m\bar{a}$" ?

Accurate to about 10^{-8} (relative error) for objects moving at "human speeds" << speed of light

much less than.

At much higher speeds, need a relativistic correction (Einstein).

(2) With respect to what should we measure acceleration for "$\sum \bar{F} = m\bar{a}$"

$\bar{a}$ is measured with respect to "inertial reference frames".

Examples of approximate inertial frames:

- earth's surface ← we will use this as an approximation
- center of earth
- center of mass of solar system
- center of " " galaxy.

more accuracy.

Acceleration due to gravity.

what is its value? $\underbrace{9.81 \ m/s^2}$

"textbook answer"

"standard value"

look up in Wolfram alpha.

g changes, depends on location on earth.

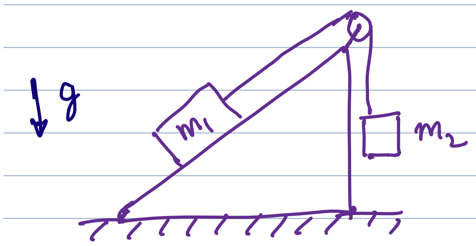
eg. Columbus. g = ? 9.82 m/s.

as low as $g = 9.79 \ m/s^2$

Example.

Given

- all surfaces are frictionless.
- pulley: massless & frictionless
- pulley rope: massless & inextensible.
- assume the tension in the rope remains constant over its whole length*



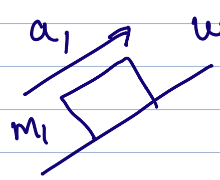
*Can be shown based on the other assumptions.

$m_1 = 10\text{ kg}$, $m_2 = 20\text{ kg}$.

$\theta = 60°$.

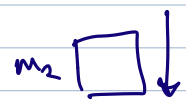
Determine accelerations of m_1 and m_2

Sign conventions

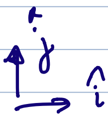


up slope $+a_1$

downward

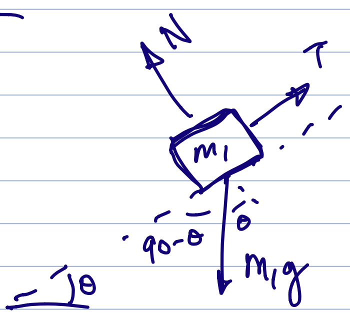


$+a_2$:

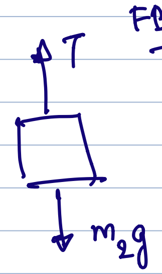


FBDs

FBD 1

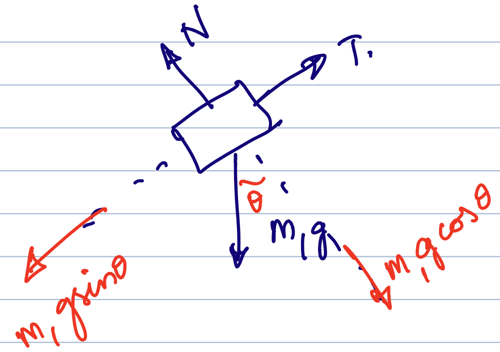


FBD 2

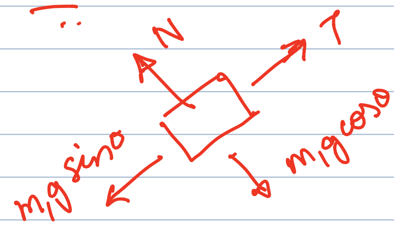


Write "F = ma" in each relevant direction for each FBD.

For FBD2: $m_2 a_2 = m_2 g - T$ (downward positive). (1)



FBD 1



Up slope. positive : $\quad m_1 a_1 = T - m_1 g\sin\theta$ ②

perpendicular to slope : $\quad \underbrace{0} = N - m_1 g\cos\theta$ ③

↑ no acceleration $\perp^{ar}$ to slope

How many unknowns : $a_1, a_2, N, T.$

3 equations, need one more equation.

Kinematics / constrained motion : $a_1 = a_2$. (relation between accelerations). ④

From ③, $\boxed{N = mg\cos\theta.}$

Use ④ in ② : $\quad m_1 a_2 = T - m_1 g\sin\theta$. ⑤

① : $\quad m_2 a_1 = m_2 g - T.$ ①

Add. $(m_1 + m_2) a_1 = \cancel{T} - m_1 g\sin\theta + m_2 g - \cancel{T}$

$= g(m_2 - m_1\sin\theta)$.

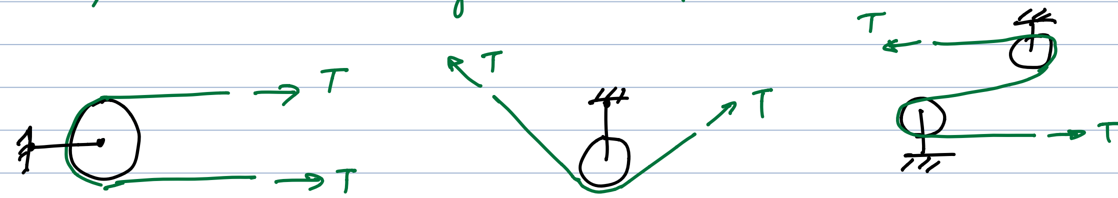
$$\Rightarrow \boxed{a_1 = \frac{(m_2 - m_1\sin\theta)\, g}{m_1 + m_2} = a_2.}$$

, Plug in numbers.

1st half of course

Note: If you have massless frictionless pulleys and massless ropes: the tension in a given rope is constant. ie., same all through the rope.

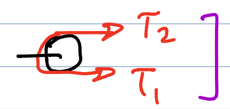
ideal pulley.



later in the course

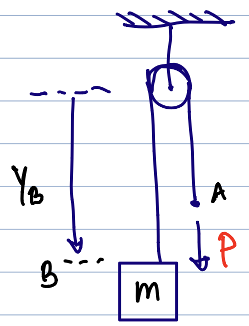
Note 1: this follows from a future topic: moment balance.

Note 2: Later in the course, we will have pulleys with masses and friction and we can no longer assume T is constant.



unequal tension.

Example



pulley: massless & frictionless ← ideal pulley

ropes: massless & inextensible. ← ideal rope.

$m = 10$ kg.

force $P = 200$ N

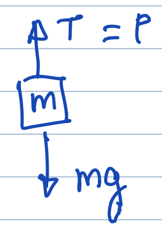
a) Determine acceleration of m and of point A.

b) Determine force exerted by pulley on ceiling.

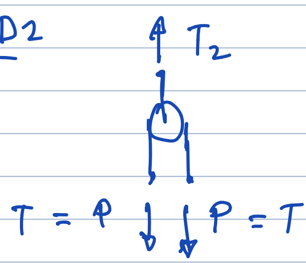
Solution:

FBD 1

Sign convention



FBD 2



$T_2 - 2P = \cancel{m_{pulley}}^{\,0} \cdot a_{pulley}$

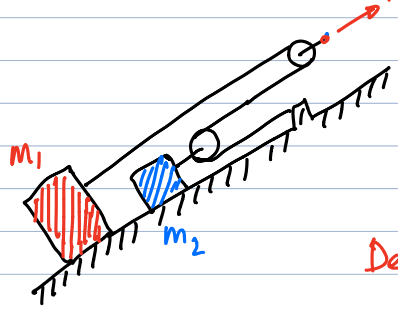
$T_2 - 2P = 0.$

$\boxed{T_2 = 2P} = \underbrace{400 \text{ N}}_{\text{ANS (b)}}$

– For FBD 1 $\therefore$ $ma = P - mg.$

$\boxed{a = \dfrac{P - mg}{m}}$ (a).

Example



$P = 200N$.

$m_1 = 10$ kg

$m_2 = 10$ kg.

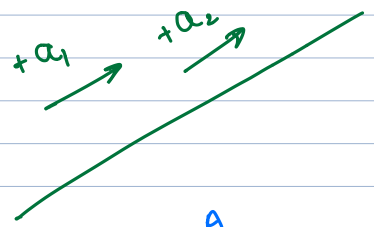
$g = 9.81\ m/s^2$

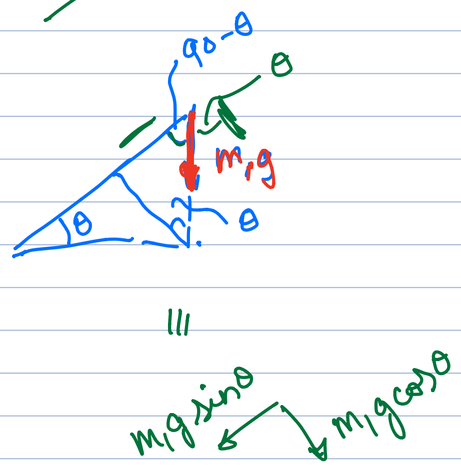
all pulleys & ropes ideal.

(massless, frictionless, inextensible)

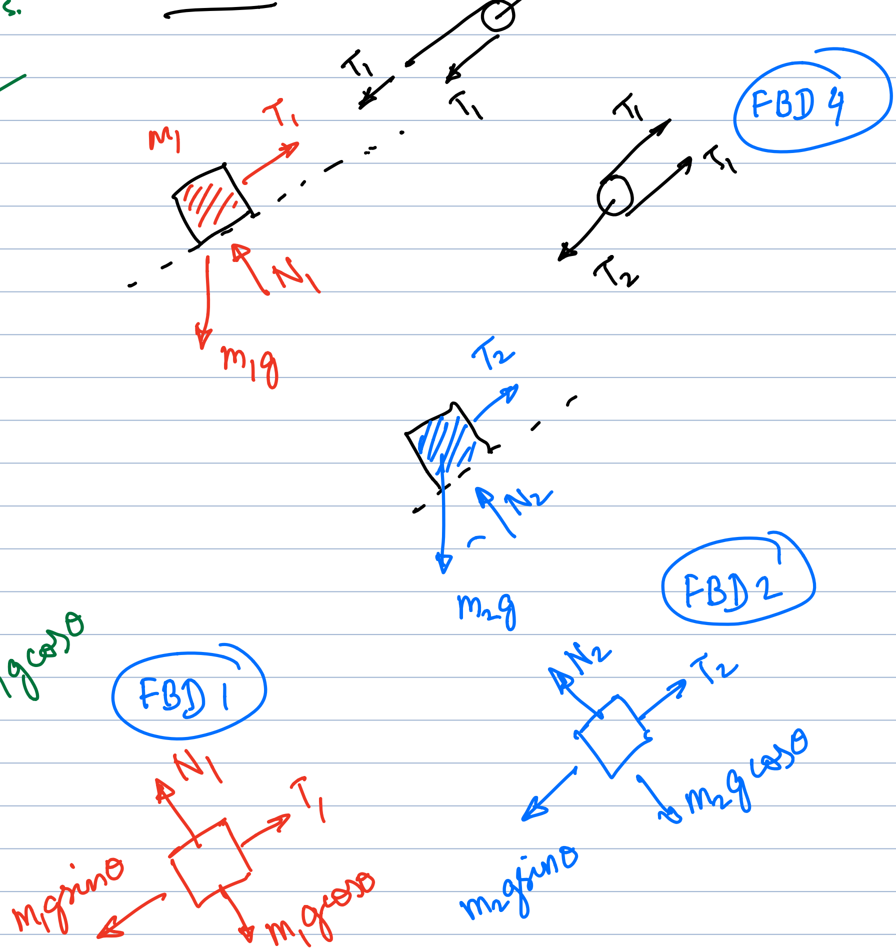
Determine a_1, a_2.

Sign conventions.





FBDs



For FBD3 (pulley)

$$P - 2T_1 = \cancel{m_{pulley,1}}^{\,0}\, a_{pulley,1} = 0$$

$\boxed{P = 2T_1}$ (1)

For FBD4

$$2T_1 - T_2 = m_{pulley_2}\, a_{p2} = 0$$

$\boxed{T_2 = 2T_1}$

(or) $\boxed{T_2 = P}$ (2)

For FBD1

up slope: $T_1 - m_1 g\sin\theta = m_1 a_1$ (3)

$\perp^{ar}$ slope: $N_1 - m_1 g\cos\theta = 0$ (0 accel.). (4)

FBD2

$T_2 - m_2 g\sin\theta = m_2 a_2$ (5)

$N_2 - m_2 g\cos\theta = 0$ (6)

$$a_1 = \frac{T_1 - m_1 g\sin\theta}{m_1} = \frac{P/2 - m_1 g\sin\theta}{m_1} = \checkmark$$

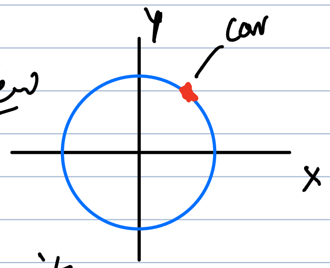
$$a_2 = \frac{T_2 - m_2 g\sin\theta}{m_2} = \frac{P - m_2 g\sin\theta}{m_2} = \checkmark$$

Example : A car is going in a circle in the horizontal plane with speed $v = 20\ m/s$ and rate of change of speed $\dot{v} = 5\ m/s^2$.

The car interacts with ground only through friction (horizontal) and normal reaction (vertical).

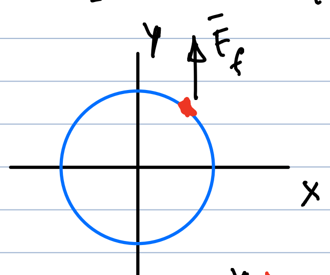
Determine the frictional force $\bar{F}_f$.

top view



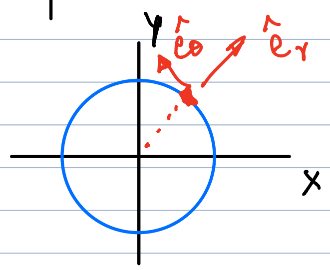
gravity is lar to plane.

top view

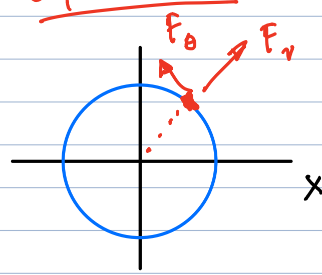


Mass $m = 1000\ kg$. radius $R = 20\ m$.

No wind resistance.



top view FBD.



$\underbrace{\Sigma \bar{F}} = m\bar{a}$ in horizontal direction.

$$F_r \hat{e}_r + F_\theta \hat{e}_\theta = m \overbrace{\left(\underbrace{-r\dot{\theta}^2 \hat{e}_r}_{\text{centripetal}} + \underbrace{r\ddot{\theta}\, \hat{e}_\theta}_{\text{tangential}} \right)}^{\text{polar } \bar{a}}.$$

Given $v = 20\ m/s$. $\quad \dot{\theta} = ?$ $\quad \bar{v} = v\hat{e}_\theta = r\dot{\theta}\hat{e}_\theta$.

$R = 20\ m$.

$$v = R\dot{\theta} \Rightarrow \dot{\theta} = \frac{v}{R} = \frac{20}{20} = \underline{\underline{1\ rad/s}}.$$

$$\dot{v} = R\ddot{\theta} \Rightarrow \ddot{\theta} = \dot{v}/R = 5/20 = \underline{\underline{0.25\ rad/s^2}}$$

$$\bar{a} = -20(1)^2 \hat{e}_r + 20 \cdot (0.25)\, \hat{e}_\theta = -20\hat{e}_r + 5\hat{e}_\theta.$$

$$\Sigma \bar{F} = F_r \hat{e}_r + F_\theta \hat{e}_\theta = 1000 \left(-20 \hat{e}_r + 5 \hat{e}_\theta \right)$$

Friction force $= \left(\underbrace{-20000 \hat{e}_r}_{\text{radial force}} + \underbrace{5000 \hat{e}_\theta}_{\text{tangential force}} \right)$ N

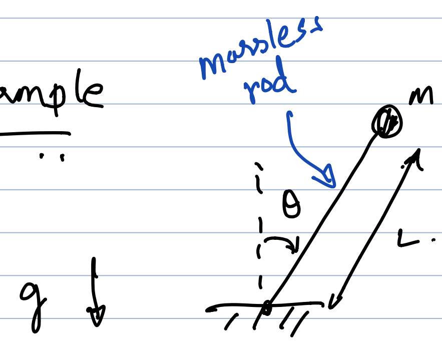
radial force ← centripetal force

tangential force ← provides increase in speed

---

the pendulum spins around

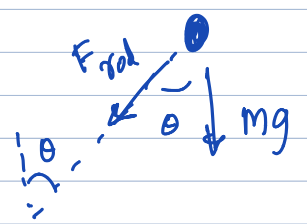
Example



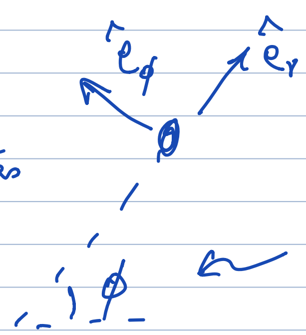
a) Determine the value of θ in terms of $\dot{\theta}$, m, g, L when the force transmitted by the rod = 0

a) Determine $\ddot{\theta}$ in terms of m, g, L, θ and possibly $\dot{\theta}$.

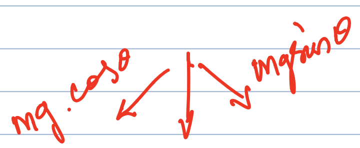
FBD



polar coordinates



← where $\phi = 90° - \theta$.



$$\bar{a} = \left(-r\dot{\phi}^2\right)\hat{e}_r + \left(r\ddot{\phi}\right)\hat{e}_\phi \quad \leftarrow \text{standard}$$

$$\Sigma \bar{F} = m\bar{a}$$

along $\hat{e}_r$: $\quad -F_{rod} - mg\cos\theta = m\left(-r\dot{\phi}^2\right) \quad (1)$

along $\hat{e}_\phi$: $\quad -mg\sin\theta = m\left(r\ddot{\phi}\right) \quad (2)$

part (a) (i) $\Rightarrow$ $-F_{rod} - mg\cos\theta = -mr\dot{\phi}^2$

$F_{rod} = 0$ (given)

$$\Rightarrow -mr\dot{\phi}^2 = -mg\cos\theta$$

$$\phi = \underbrace{90-\theta}_{deg} = \frac{\pi}{2} - \theta \text{ rad}. \quad \boxed{\dot{\phi} = -\dot{\theta}} \Rightarrow \dot{\phi}^2 = (-\dot{\theta})^2 = \dot{\theta}^2$$

$$\cancel{-}\cancel{m}r\dot{\theta}^2 = \cancel{-}\cancel{m}g\cos\theta$$

ANS(a).

$$\dot{\theta}^2 = \frac{g}{r}\cos\theta \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{r\dot{\theta}^2}{g}\right)}$$

(b) (2) $\Rightarrow$ $\cancel{m}r\ddot{\phi} = -\cancel{m}g\sin\theta$

$$r\ddot{\phi} = -g\sin\theta.$$

$$\phi = \frac{\pi}{2} - \theta \Rightarrow \dot{\phi} = -\dot{\theta} \Rightarrow \ddot{\phi} = -\ddot{\theta}$$

$$\Rightarrow r(-\ddot{\theta}) = -g\sin\theta.$$

$$\boxed{\ddot{\theta} = \frac{g}{r}\sin\theta}$$

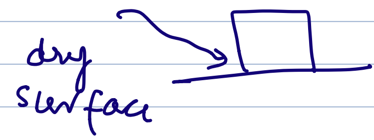
ANS (b).

Today: **TWO TOPICS**.

Coulomb friction &

Work - Energy principles

Friction
- viscous friction
- dry friction



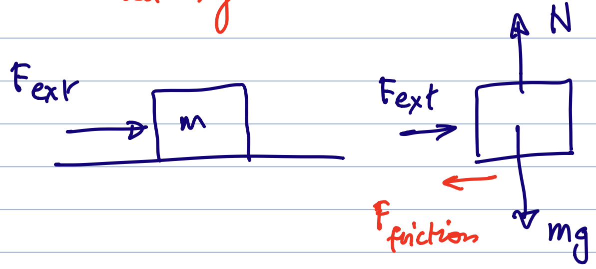
→ Coulomb's laws

Coulomb friction.

Coulomb friction

- static friction
- kinetic friction

Consider a block sitting on a table. Push with F_{ext}.



what is $F_{friction}$?

For small enough F_{ext}, the mass m does not slip.

("slip" ≡ relative velocity between 2 contacting surfaces).

$$F_{friction} = F_{ext}$$

as long as $|F_{friction}| \leq \mu_s N$.

$\mu_s N$ is the "static friction": maximum friction force possible between the 2 surfaces.

μ_s = static friction coefficient.

(or) coefficient of static friction.

In this example, if $F_{ext} > |\mu_s N|$, then the mass starts slipping.

when there is slip (relative motion between contacting surfaces), the friction force has magnitude $|F_{friction}| = \mu_k N$

and is directed opposite the relative velocity.

μ_k. Kinetic friction coefficient.

More generally :

If there is slip, $|\bar{F}_{friction}| = \mu_k N$

and opposes slip velocity.

If there is no slip, $F_{friction}$ = whatever it takes to avoid slip as long as

$$|F_{friction}| \leq \mu_s N.$$

Will it slip? When does it slip?

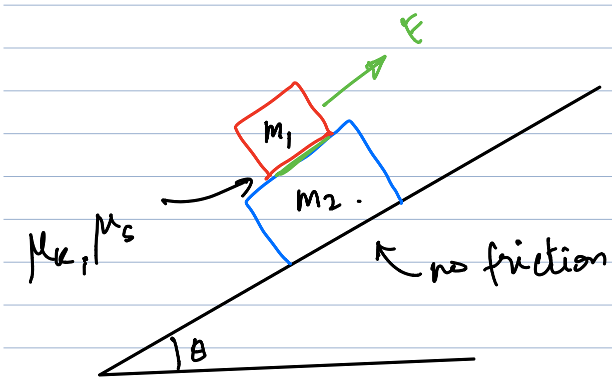
(1) Assume there is no slip.

(2) Compute friction force using $\Sigma F = ma$, etc.
(NOT using μN). Can assume $\bar{F}_{friction}$ is ANY direction.

(3) If computed magnitude $|\bar{F}_{friction}| \leq \mu_s N$, no slip

else there is slip.

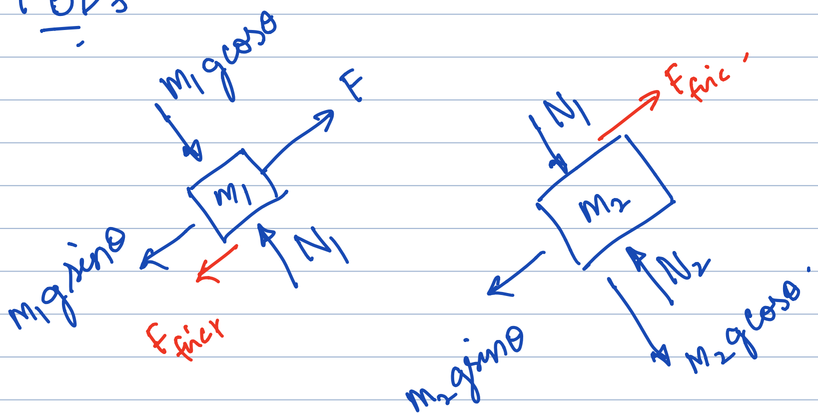
Example:



Given.
- Two blocks m_1, m_2.
- No friction between m_2 and slope.
- μ_s and μ_k are the static & kinetic friction coefficients between m_1 and m_2.
- External force F applied.

Under what conditions will m_1 slip on m_2?

FBDs:



Assume no slip. & find F_{fric} &

$+a$ ↗

No slip $\Rightarrow$ Both masses have same acceleration a.

$$F - m_1 g\sin\theta - F_{fric} = m_1 a. \quad (1)$$

$$F_{fric} - m_2 g\sin\theta = m_2 a \quad (3)$$

$$N_1 - m_1 g\cos\theta = 0 \quad (2)$$

$$N_2 - N_1 - m_2 g\cos\theta = 0. \quad (4)$$

4 eqns : 4 unknowns N_1, N_2, F_{fric}, a. SOLVE.

$\rightarrow$ Find F_{fric} and N_1.

Then, condition for No SLIP : $|F_{fric}| < \mu_s N_1$. ANS

condition for SLIP : $|F_{fric}| > \mu_s N_1$.

Solving the equations :

$$(2) \Rightarrow N_1 = m_1 g\cos\theta \quad (5).$$

To solve for F_{fric}, need to eliminate 'a' from (1) and (3).

Do $(1)/m_1$ and $(3)/m_2$:

$$(1)/m_1 : \quad \frac{F - m_1 g\sin\theta}{m_1} - \frac{F_{fric}}{m_1} = a$$

$$(3)/m_2 : \quad \frac{F_{fric} - m_2 g\sin\theta}{m_2} = a.$$

Subtract ①$/m_1$ − ③$/m_2$:

$$\frac{F - m_1 g\sin\theta}{m_1} - \frac{F_{fric}}{m_1} - \frac{F_{fric}}{m_2} + \frac{\cancel{m_2} g\sin\theta}{\cancel{m_2}} = a - a = 0$$

$$\frac{F}{m_1} - \cancel{g\sin\theta} - F_{fric}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \cancel{g\sin\theta} = 0$$

$$F/\cancel{m_1} = F_{fric}\left(\frac{m_1 + m_2}{\cancel{m_1} m_2}\right).$$

$$\Rightarrow \boxed{F_{fric} = m_2 F/(m_1 + m_2)}$$

So condition for slip:

$$|F_{fric}| > \mu_s N_1 .$$

$$\Rightarrow \boxed{\left|\frac{m_2 F}{m_1 + m_2}\right| > \mu_s m_1 g\cos\theta}$$

condition for SLIP.

ANS

General remark: If the friction force is, say a vector in 2D or 3D like

$$\bar{F}_{fric} = F_x\hat{i} + F_y\hat{j} \text{ or } F_r\hat{e}_r + F_\theta\hat{e}_\theta .$$

Then no slip when $|\bar{F}_{fric}| = \sqrt{F_x^2 + F_y^2} = \sqrt{F_r^2 + F_\theta^2} < \mu_s N$.

MAGNITUDE

Work and Energy

Important terms

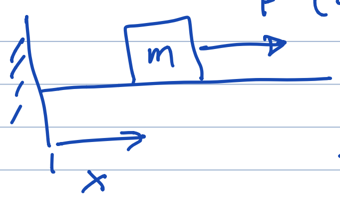
Work : mechanical work of a force

Energy (mechanical) $\left\{\begin{array}{l}\text{kinetic energy} \\ \text{potential energy} \left\{\begin{array}{l}\text{gravitational} \\ \text{elastic} \\ \text{other things.}\end{array}\right.\end{array}\right.$

$$\text{power} = \frac{d\,\text{Work}}{dt}$$

Example :

F (sum of all forces)



$$F = ma = m\ddot{x} = \frac{dv}{dt}.$$

$$F = \frac{m\,dv}{dt}.$$

$$Fv = mv\frac{dv}{dt}.$$

$$\int Fv\,dt = \int mv\,dv$$

$$\int_{x_i}^{x_f} F\frac{dx}{\cancel{dt}}\,\cancel{dt} = \int_{x_i}^{x_f} F\,dx = m\int_{v_i}^{v_f} v\,dv = m\left[\frac{v^2}{2}\right]_{v_i}^{v_f} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$\int_{x_i}^{x_f} Fv\,dt = \boxed{\underbrace{\int_{x_i}^{x_f} F\,dx}_{} = \underbrace{\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2}_{}.} \quad (1)$$

Define : Work done by Force F = Change in kinetic energy.

Define : KE of the particle $= \frac{1}{2} m v^2$.

(1) is called the Work-energy theorem.

"Work done by all forces on an FBD" = Total change in KE of all objects in the FBD.

From (1), we see work $= \int F\,dx = \int \underbrace{(Fv)}\,dt = \int \frac{dW}{dt}\,dt$

Define: power

"mechanical power".

$\frac{d\text{Work}}{dt}$

In 2D or 3D:

dot product

Mechanical Work =

(scalar quantity)

$\int \underbrace{(\bar{F} \cdot \bar{v})}_{\text{power} = \bar{F}\cdot\bar{v}}\,dt = \int \bar{F} \cdot d\bar{x}$.

Kinetic energy of a particle $= \frac{1}{2} m |\bar{v}|^2 = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2\right)$

in 2D: $\frac{1}{2} m \left(v_x^2 + v_y^2\right) = \frac{1}{2} m \left(v_r^2 + v_\theta^2\right)$

$= \frac{1}{2} m \left(v_t^2 + v_n^2\right)$.

If the force is a constant . .

$$W = \int_{x_i}^{x_f} F\,dx = F\int_{x_i}^{x_f} dx = F\,(x_f - x_o).$$

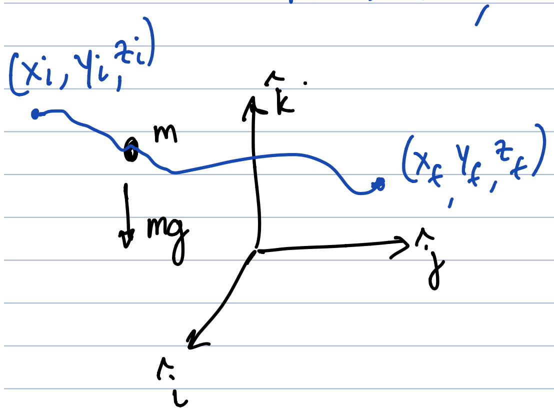
$= \text{force} \times \text{displacement}.$ ONLY true if F constant.

Not true in general.

in 2D/3D. if $\bar{F}$ is constant,

$$W = \bar{F}\cdot(\bar{r}_f - \bar{r}_i)$$

Example: Determine work done by gravity on particle mass m, when it goes from position $\bar{r}_i$ to $\bar{r}_f$.



$$\bar{F}_{gravity} = -mg\hat{k} = \text{constant}$$

$$\text{Work} = \int \bar{F}\cdot d\bar{x} = \bar{F}\cdot(\bar{r}_f - \bar{r}_i)$$

$$= (-mg\hat{k})\cdot(\bar{r}_f - \bar{r}_i).$$

$$\bar{r}_f = x_f\hat{i} + y_f\hat{j} + z_f\hat{k}, \quad \bar{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}.$$

$$\text{Work} = (-mg\hat{k})\cdot\left[(x_f - x_i)\hat{i} + (y_f - y_i)\hat{j} + (z_f - z_i)\hat{k}\right]$$

$$= 0 + 0 - mg\,(z_f - z_i)$$

$$\text{Work by gravity} = -mg\,(z_f - z_i) = mg z_i - mg z_f. \quad (1)$$

Define: potential energy $= mgz$

Work done by gravity $= PE_i - PE_f$.

$= -(PE_f - PE_i)$.

does not depend on path. ONLY final & initial!

"Potential energy" is just a convenient way of writing work done by some forces. called "conservative forces". eg. gravity, springs, for which work done = function of only final & initial positions, and not on path.

Work done by conservative forces $= PE_i - PE_f$.

Non-conservative forces: friction, other external forces. whose work done needs to be computed explicitly via integration (may depend on path).

Work energy theorem:

Work done by all forces = total change in KE

$= KE_f - KE_i$

if all forces are conservative then work done $= PE_i - PE_f$.

So, $PE_i - PE_f = KE_f - KE_i$

(or) $\boxed{PE_i + KE_i = PE_f + KE_f}$ (*)

We call "PE + KE" = total mechanical energy.

So, (*) is called "conservation of total (mechanical) energy".

If only some forces are "conservative."

then the work energy theorem is

Work done by all forces } = Change in KE

Work done by conservative forces + Work done by other forces (not gravity or springs). = change in KE

$$PE_i - PE_f + W_{i\to f} = KE_f - KE_i$$

Rearranging,

$$\boxed{PE_i + KE_i + W_{i\to f} = PE_f + KE_f}$$

KEY RELATION

total initial energy + work done by other forces = total final energy.

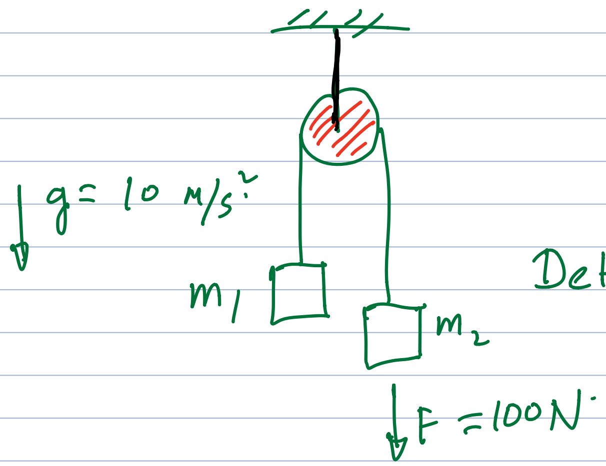
Another version : (useful for problems)

$$(KE_f - KE_i) + (PE_f - PE_i) = \text{work done by other forces.}$$

(or) $$\underbrace{(KE_f + PE_f)}_{\text{total final energy}} - \underbrace{(KE_i + PE_i)}_{\text{total initial energy}} = \text{work done by other forces.}$$

non conservative.

Example



Ideal pulley and ropes.

$m_1 = 10$ kg. $m_2 = 20$ kg.

$F = 100$ N.

Masses initially at rest.

Determine the downward velocity of m_2 when m_2 has fallen by 0.1 m. ["final"]

Let's use the work-energy version :

(1) $$(KE_f - KE_i) + (PE_f - PE_i) = \text{work done by other forces.}$$

$KE_i = 0$ (starting at rest). (2)

$$KE_f = \frac{1}{2} m_1 v_{1_f}^2 + \frac{1}{2} m_2 v_{2_f}^2 ,$$

where v_{1f} and v_{2f} are final velocities of the masses. If we assign sign conventions

$v_{1f}\uparrow \quad v_{2f}\uparrow$ then if $v_{1f} = v$

$v_{2f} = -v$.

(3) So $KE_f = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 (-v)^2$

$= \frac{1}{2}(m_1 + m_2) v^2 . = \frac{1}{2}(10+20) v^2 = 15 v^2.$

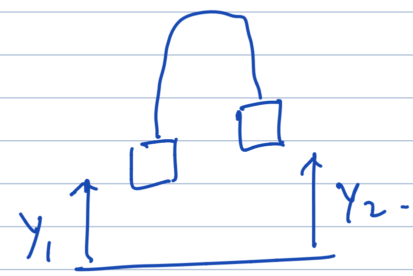
Work done by F = Force . displacement

because $F = \text{const}$

$= (100)(0.1) = 10 \text{ J}.$

$PE_f - PE_i = ?$

gravity



$= m_2 g (y_{2f} - y_{2i}) + m_1 g (y_{1f} - y_{1i})$

$= m_2 g \underbrace{(-0.1 \text{ m})}_{y_2 \text{ goes down}} + m_1 g \underbrace{(+0.1 \text{ m})}_{y_1 \text{ goes up.}}$

$= (20)(1\cancel{0})(-\cancel{0.}1) + (10)(1\cancel{0})(\cancel{0.}1)$

$= -20 + 10 . = -10$ (4)

$PE_f - PE_i = -10$ (4)

Put everything in (1)

$(KE_f - KE_i) + (PE_f - PE_i) = $ work by F

$(15v^2 - 0) + (-10) = 10$

$15v^2 = 20$

$v^2 = \frac{20}{15} \Rightarrow v = \pm\sqrt{\frac{20}{15}}$ m/s.

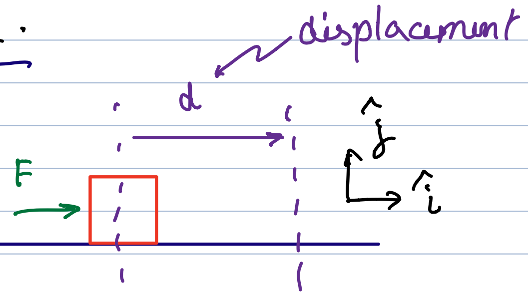
Sign picked so that m_2 goes down &

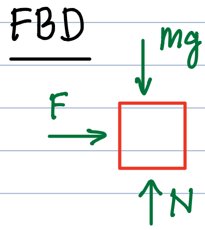
m_1 goes up.

$$\boxed{\begin{aligned} v_{2f} &= -\sqrt{\frac{20}{15}} \text{ m/s.} \\ v_{1f} &= +\sqrt{\frac{20}{15}} \text{ m/s.} \end{aligned}}$$

ANS

Example.





Work done by $F = Fd$.

Work done by $mg = mg(0) = 0$

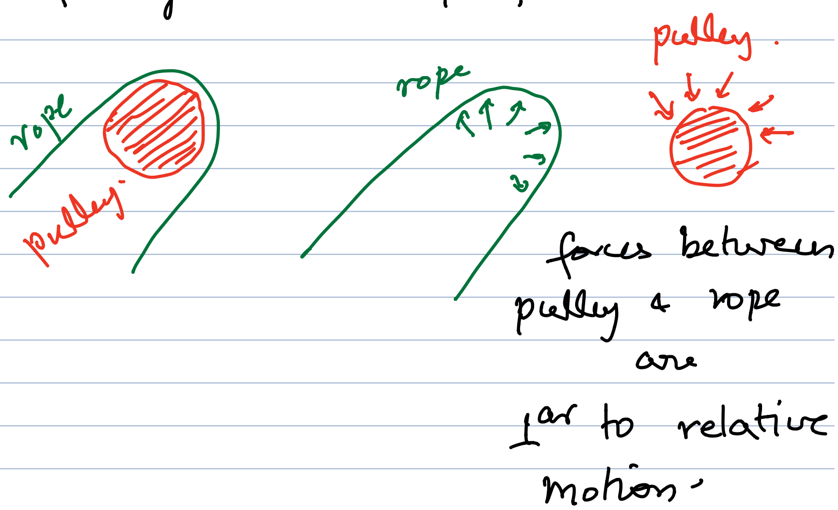
More precisely, work by $-mg\hat{j} = (-mg\hat{j}) \cdot (d\hat{i})$

$$= 0.$$

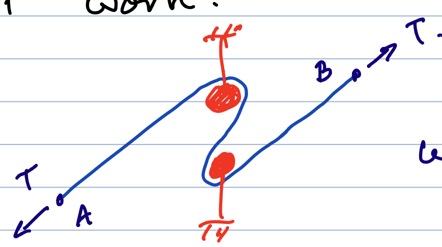
Work by $N\hat{j} = (N\hat{j}) \cdot (d\hat{i}) = 0$.

Work $= 0$ when force perpendicular to motion of point of application of force.

Remarks : Similarly, ideal pulleys do not perform work on ideal ropes



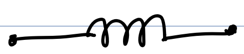
Similarly, ideal ropes connected to masses on either end perform (tension in) no net work.



The tension at point A performs equal & opposite work to tension at point B. They cancel!

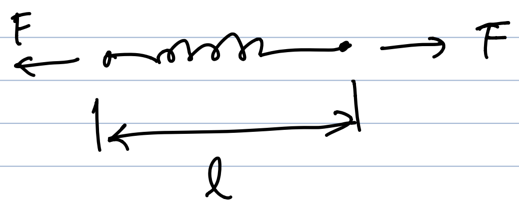
If the ropes and pulleys are not "ideal", eg. friction + mass. there is work done by them, often negative. "transmission losses". (Same for gears in cars).

Springs.



① They have a "rest length", also called "free length", "stress free length", "unstretched length". Let us call this l_0. (NOT INITIAL LENGTH).

When they are at this length, force transmitted by them = 0. (tension = 0).



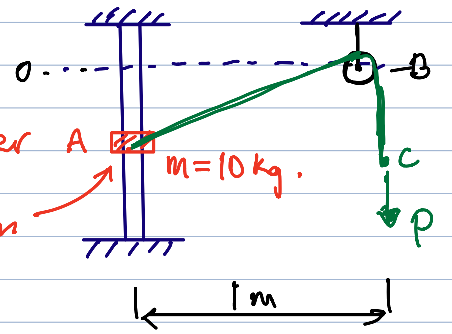
$$F = k \overbrace{(l - l_0)}^{\text{stretch from } l_0}.$$

Spring stiffness — current length — unstretched or stress free length.

A stretched or compressed string has potential energy $PE = \frac{1}{2} k (l - l_0)^2$.

(same PE for both $l - l_0 > 0$ and < 0 because of the square).

Example



$g = 9.81$

OB is a horizontal line. B is pulley center. The slider A starts 2m below the level OB and is pulled up by force $P = 1000N$, starting from rest. Find the velocity of the slider when it reaches level OB.

Assume pulley radius is negligible.

Work-energy theorem.

$$PE_i + KE_i + \text{work by force } P = PE_f + KE_f$$

$KE_i = 0$

$KE_f = \frac{1}{2} m v_f^2$

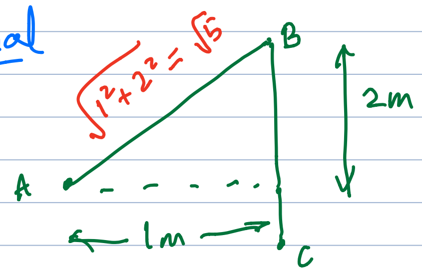
$PE_f - PE_i = mg(2m)$. ← it is often simpler to write $PE_f - PE_i$ than individual PE for gravity.

↑ the amount by which the mass rose.

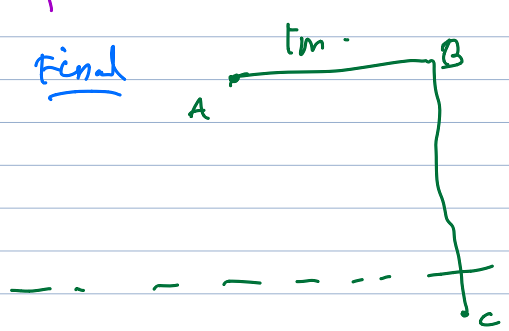
Work by force P = Force P · displacement of point C. (because P = constant).

By how much does point C fall?

Initial



Final



Because $|AB| + |BC| =$ length constant

$$|AB|_i + |BC|_i = |AB|_f + |BC|_f$$

$$|BC|_f - |BC|_i = |AB|_i - |AB|_f$$

$$= \sqrt{5} - 1$$ ← distance by which C drops.

$\Rightarrow$ Work by P $= 1000(\sqrt{5}-1)$.

Putting everything together

$$PE_f - PE_i + KE_f - KE_i = \text{Work by } P$$

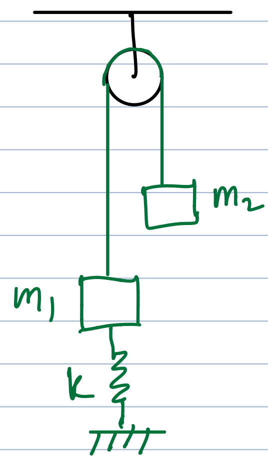
$$mg(2) + \frac{1}{2}mv_f^2 - 0 = 1000(\sqrt{5}-1).$$

$$(1\cancel{0} \times 9.81 \times 2) + \frac{1}{2} \cdot 1\cancel{0} \cdot v_f^2 = 100\cancel{0}(\sqrt{5}-1).$$

$$v_f = \text{solve.} = \sqrt{\left(100(\sqrt{5}-1) - 19.62\right)2}$$

$$= \text{whatever.}$$

Example :



$m_1 = 1kg$

$m_2 = 2kg$.

$K = 1000$ N/m.

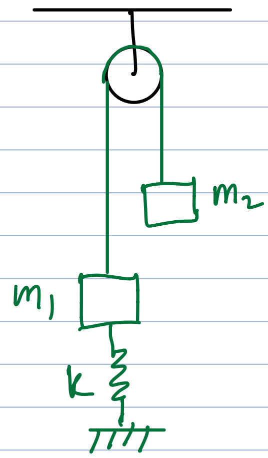
V_{m_1} initial $= 0.1$ m/s upwards.

initial stretch of spring relative to unstretched $= s_i = 0.05$ m.

Determine max stretch of spring (when the masses momentarily stop).

– Say $= s_f$.

Example :



$m_1 = 1kg$

$m_2 = 2kg.$

$k = 1000\ N/m.$

V_{m_1} initial $= 0.1\ m/s$ upwards.

$g = 10\ m/s^2$

initial stretch of spring relative to unstretched $= S_i = 0.05\ m.$

Determine max stretch of spring S_f (when the masses momentarily stop).

Work-energy balance:

$$PE_i + KE_i + \text{Work by other forces} = PE_f + KE_f.$$

$$(PE_f - PE_i) + (KE_f - KE_i) = \text{Work by other forces.}$$

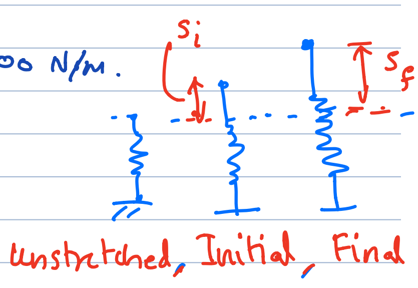
$KE_i = ?$ Given: $V_{m_1} \uparrow 0.1\ m/s.$ $V_{m_2} \downarrow 0.1\ m/s.$

$$= \frac{1}{2} m_1 (0.1)^2 + \frac{1}{2} m_2 (0.1)^2 = \frac{1}{2} 1 \cdot (0.1)^2 + \frac{1}{\cancel{2}} \cancel{2} (0.1)^2.$$

$$= \frac{0.01}{2} + 0.01 = 0.005 + 0.01 = 0.015\ J$$

$KE_f = 0$ (when the stretch is maximum, the masses just come to rest).

If S_f = final stretch, $S_i = 0.05$ = initial stretch, $k = 1000\ N/m.$



$$PE_f - PE_i \text{ (spring)} = \frac{1}{2} k\, S_f^2 - \frac{1}{2} k\, S_i^2$$

$$= 500\, S_f^2 - 500\,(0.05)^2$$

$PE_f - PE_i$ gravity : $\underbrace{m_1 g (s_f - s_i)}_{\text{mass 1 goes up by } s_f - s_i \text{ as the spring stretches}} + \underbrace{m_2 g \left[- (s_f - s_i) . \right]}_{\text{mass 2 goes down by } s_f - s_i \text{, hence the negative sign.}}$

$$= 1 (10)(s_f - 0.05) + 2(10)(-s_f + 0.05)$$

$$= -0.5 + 10 s_f - 20 s_f + 1 = -10 s_f + 0.5 .$$

Work done by 'other' forces $= 0$ (nothing but gravity & springs, which are accounted for in PE).

Putting everything together,

$$(PE_f - PE_i) + (KE_f - KE_i) = \text{work by other forces}$$

$$\underbrace{500 s_f^2 - 500(0.05)}_{\text{spring}} \underbrace{- 10 s_f + 0.5}_{\text{gravity}} + 0.015 = 0 \quad — (1)$$

Solve equation (1) for s_f. Quadratic equation.

REVIEW FOR MIDTERM 1

HW1 : 1D motion :

Differentiate: position $\rightarrow$ velocity $\rightarrow$ acceleration.

Integrate: accelerate $\rightarrow$ velocity $\rightarrow$ position.

If a is function of t: $v - v_0 = \int_0^t a(t)\,dt$

If a is function of s: $\int v\,dv = \int a(s)\,ds.$

If a is function of v : $\int \frac{dv}{a(v)} = \int dt.$

If v is function of t : $\int ds = \int v(t)\,dt.$

If v is function of s : $\int \frac{ds}{v(s)}$

HW2 : Coordinate systems.

Cartesian : $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k},$

$\bar{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}.$

$\bar{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}.$

Tangential - normal: $\bar{v} = v\,\hat{e}_t$

$\bar{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n.$

Polar : $\bar{r} = r\hat{e}_r$

$\bar{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta.$

$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$

Conversions:

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}.$$
$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}.$$

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$
$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta.$$

HW3: relative motion

$$\bar{V}_{A/B} = \bar{V}_A - \bar{V}_B \quad , \quad \bar{a}_{A/B} = \bar{a}_A - \bar{a}_B.$$

$$\bar{V}_{A/B} = \bar{V}_{A/C} - \bar{V}_{B/C} \quad , \quad \bar{a}_{A/B} = \bar{a}_{A/C} - \bar{a}_{B/C}.$$

Constrained motion: eg., relation between velocities of different objects or different direction due to constraints.

eg. masses connected by inextensible ropes in pulley systems.

HW4:

- FBD diagrams (for each body, typically)
- Sign conventions. & coordinate system. cartesian / polar / tangential-normal.
- $\Sigma \bar{F} = m\bar{a}$ for each FBD.
  - write vector equation or
  - write scalar equations in $\perp^{ar}$ directions
- possibly use constrained motion (length of pulley ropes) to get an additional equation.

SOLVE the equations for the relevant quantities.

HW 5

$$PE_i + KE_i + \text{Work by other forces} = PE_f + KE_f.$$

$$(PE_f - PE_i) + (KE_f - KE_i) = \text{work by other forces.}$$

$$\text{Work} = \int \bar{F} \cdot d\bar{r} = \int (\bar{F} \cdot \bar{v})\, dt$$

If $\bar{F}$ = constant, work $= \bar{F} \cdot \bar{d}$.

PE

gravity $= mgy$ (y measured upward)

spring $= \frac{1}{2} k \underbrace{(l - l_0)}^2 = \frac{1}{2} k (\text{stretch})^2$

change in length measured from the unstretched or the force-free length l_0.

LINEAR IMPULSE AND MOMENTUM

For a single particle,

$$\Sigma \bar{F} = m\bar{a} \, .$$

Integrate both sides with respect to time

$$\int_{t_i}^{t_f} (\Sigma \bar{F})\, dt = \int_{t_i}^{t_f} (m\bar{a})\, dt = \left[m\bar{v} \right]_{t_i}^{t_f} = m\bar{v}_f - mv_i$$

Summarizing

$$\boxed{\underbrace{\int_{t_i}^{t_f} (\Sigma \bar{F})\, dt}_{\text{Called "Impulse" of force } \Sigma \bar{F}} = \underbrace{m\bar{v}_f - m\bar{v}_i}_{\text{change in linear momentum for the particle.}}}$$

vector equation (1)

(Momentum $= \bar{G} = m\bar{v}$.).

This equation is called the Impulse - momentum relation.

The above relation was derived for a single particle.

But it is true for systems with multiple masses or particles. For multiple particles,

$$\Sigma \bar{F} = m_1 \bar{a}_1 + m_2 \bar{a}_2 + \ldots$$

sum of all external forces

momentum for all masses.

$$\Rightarrow \int_{t_i}^{t_f} \Sigma \bar{F}\, dt = \underbrace{\left(m_1 \bar{v}_{1_f} + m_2 \bar{v}_{2_f} + \ldots \right)}_{\text{final momentum}} - \underbrace{\left(m_1 \bar{v}_{1_i} + m_2 \bar{v}_{2_i} + \ldots \right)}_{\text{initial momentum}}. \quad (2)$$

Total impulse = Change in total linear momentum $= \Delta \bar{G}$.

Special case 1. If $\Sigma \bar{F} = \bar{0}$ (no force in any direction)

then, Impulse $= \int \Sigma \bar{F}\, dt = \bar{0}$ over any time period.

$\Rightarrow$ Change in total linear momentum $= \bar{0}$.

vee

3 $$m_1 \bar{v}_{1f} + m_2 \bar{v}_{2f} + \dots m_N \bar{v}_{Nf} = m_1 \bar{v}_{1i} + m_2 \bar{v}_{2i} + \dots + m_N \bar{v}_{Ni}$$

Arrh

Definition : If $\bar{r}_j$ is position vector of a mass m_j

then center of mass position

$$\bar{r}_{COM} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2 + \dots + m_N \bar{r}_N}{m_1 + m_2 + \dots + m_N}.$$

$$\bar{v}_{COM} = \frac{m_1 \bar{v}_1 + m_2 \bar{v}_2 + \dots}{m_1 + m_2 + \dots}$$ velocity of center of mass.

$\Rightarrow$ Total momentum $= (m_1 + m_2 + \dots)\, \bar{v}_{COM}$

$= \text{total mass} \cdot \bar{v}_{COM}$.

$\Rightarrow$ Eqn 3 can be rewritten as

$$(\cancel{\text{total mass}})\, \bar{v}_{COM_f} = (\cancel{\text{total mass}})\, \bar{v}_{COM_i}.$$

$$\bar{v}_{COM_f} = \bar{v}_{COM_i}$$

rel. to an inertial frame.

If there are no external forces,

the center of mass velocity does not change.

— can not change —.

Example.

← astronaut in outerspace. far from gravity.
$\sum \bar{F} = \bar{0}$ on them.

1) they cannot change their $\bar{V}_{COM}$.

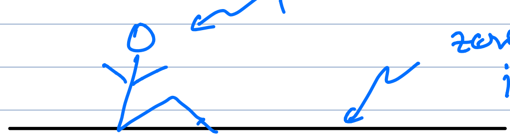
2) If they start with $\bar{V}_{COM} = \bar{0}$, remains zero for all time! (scary).

How to move? throw a shoe = same principle as a rocket, which throws gas.

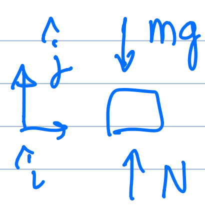
COM of (shoe + person) remains fixed! while each moves in opposite directions

Special case 2

$\sum \bar{F} = \bar{0}$ in some directions



$\sum \bar{F} = \bar{0}$ along $\hat{i}$ because no friction



along $\hat{j}$: $\sum \bar{F} \cdot \hat{j} = N - mg$.

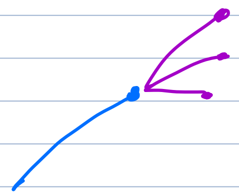
if $N > mg$, you can stand up or jump.

Momentum conserved along $\hat{i}$

but not $\hat{j}$, because you can push against the ground to make $N > mg$.

Example 3 : "Angry bird problem".

A single mass m (10 kg) breaks up into 3 pieces m_1 (2 kg), m_2 (3 kg) and m_3 (5 kg).



Just before breakup, the velocity is

$\bar{v}_i = 10$ m/s at $45°$ to horizontal $\angle 45°$

Just after breakup, the velocities are :

for m_1 : $\bar{v}_{1_f} = 10$ m/s at $60°$ to horizontal. $\angle 60°$

for m_2 : unknown . Determine !

for m_3 : $\bar{v}_{3_f} = 10$ m/s at $-60°$ to horizontal $60°$

Finally, you are told that the breakup happens in infinitesimal time. (arbitarily small duration $\to 0$).

Solution :

Impulse = change in total momentum.

$$\text{Impulse} = \int_{t_i}^{t_f} \Sigma \bar{F}\, dt \;=\; \int_{t_i}^{t_f} (\text{gravity} + \text{drag})\, dt .$$

$$\int_{t_i}^{t_f} -mg\,\hat{j} = -mg\,(t_f - t_i)\,\hat{j} \to 0 \quad \text{as} \quad t_f - t_i \to 0$$ (infinitesimal time)

Impulse $= \bar{0}$ because we are integrating finite (not infinite) forces over infinitesimal time (approaching zero).

So $\bar{0}$ = change in total momentum.

Initial momentum = final momentum.

$$m\bar{v}_i = m_1 \bar{v}_{1_f} + m_2 \bar{v}_{2_f} + m_3 \bar{v}_{3_f} .$$

$$10 \cdot \left(10\cos 45°\,\hat{i} + 10\sin 45°\,\hat{j}\right) = 2 \cdot \left(10\cos 60°\,\hat{i} + 10\sin 60°\,\hat{j}\right)$$
$$+\, 3 \cdot \bar{v}_{2_f} + 5\left[10\cos(-60°)\,\hat{i} + 10\sin(-60°)\,\hat{j}\right]$$

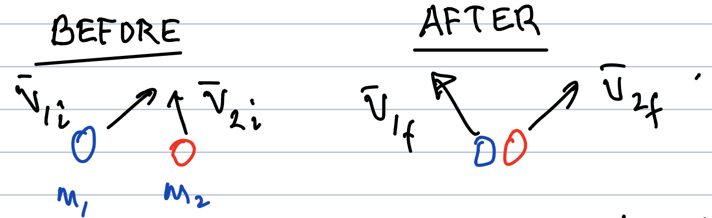
$$70.71\hat{i} + 70.71\hat{j} = 10\hat{i} + 17.32\hat{j} + 3\bar{v}_{2f} + 25\hat{i} - 43.30\hat{j}.$$

$$35.71\hat{i} + 96.69\hat{j} = 3\bar{v}_{2f}$$

$$\boxed{\bar{v}_{2f} = 11.90\hat{i} + 32.23\hat{j}}$$

ANS

COLLISIONS



Usually takes place in small duration Δt.

As discussed last lecture, for the two bodies taken together, $\Sigma \bar{F}$ may be gravity, drag, friction.

$$\Sigma \bar{F} = \text{finite}$$

with $t_f - t_i \to 0$ and $\Sigma \bar{F} = $ finite

$$\int_{t_i}^{t_f} \Sigma \bar{F}\, dt \to 0$$

Recall, Impulse = Total Final momentum − Total Initial momentum

$\Rightarrow$ 0 = Change in momentum

Momentum is conserved. ← not because $\Sigma \bar{F} = \bar{0}$ but because $\int_{t_i}^{t_f} \Sigma \bar{F}\, dt \to 0$ as $t_f \to t_i$

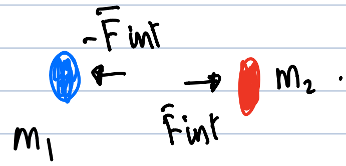
What about the internal forces between the colliding masses?

Say $\bar{F}_{int}$ is the force of m_1 on m_2 during collision.

and $-\bar{F}_{int}$ is the force of m_2 on m_1.

↳ internal reaction force.

During collision.



We can apply impulse momentum or $\bar{F} = m\bar{a}$ for each mass.

m_2 goes from $\bar{v}_{2i}$ to $\bar{v}_{2f}$ in $\Delta t = t_f - t_i \to 0$.

- if $\bar{v}_{2f} - \bar{v}_{2i}$ is NON-ZERO.

then during Δt, the mean acceleration

$$\bar{a}_2 = \frac{\bar{v}_{2f} - \bar{v}_{2i}}{\Delta t} \to \infty \quad \text{as } \Delta t \to 0.$$

Finite change in $\bar{v}$ in $\Delta t \to 0 \Rightarrow \bar{a} \to \infty$.

$\bar{a} \to \infty \Rightarrow \bar{F}_{int} \to \infty$ (the internal force).

In typical collisions, $\bar{F}_{int} \to$ large.

idealized as $\bar{F}_{int} \to \infty$ as $\Delta t \to 0$.

$$\text{Impulse of } \bar{F}_{int} = \underbrace{\int_{t_i}^{t_f} \bar{F}_{int}\, dt}_{} = m_2(\bar{v}_{2f} - \bar{v}_{2i}).$$

∞ times $0 \to$ finite change in momentum

Finite impulse from infinite forces over infinitesimal time.

Is total mechanical energy conserved over the collision?

Not generally. Collisions *may* have Δ mechanical energy $\neq 0$. (usually < 0).

KE $\rightarrow$ heat, sound, mechanical damage, etc.

Explosions are like reverse of collisions.
(eg. internal chemical or nuclear energy $\hookrightarrow$ KE, so $\Delta KE > 0$).

Elastic collisions.

Δ mechanical energy $= 0$

Note: for collisions with $\Delta t \to 0$,
Δ mechanical energy $= \Delta KE$
because $\Delta PE \to 0$ as $\Delta t \to 0$.
(no change in position in $\Delta t \to 0$).
so no change in PE.

Types of collisions.

Elastic : $\Delta KE = 0$

Inelastic : $\Delta KE < 0$ (some lost energy).

Plastic collision or perfectly inelastic

$\Rightarrow$ The two particles have same velocity after collision : "stuck together"

$(\bar{v}_{2f} = \bar{v}_{1f})$.

The maximum amount of KE that can be lost in a collision is if the collision was a plastic collision.

(Note: because of momentum conservation, one cannot generally lose ALL the energy in a collision between 2 free-moving particles, unless we start with zero momentum).

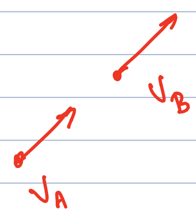
Coefficient of restitution for particle collisions

$e = 1$: elastic collision

$e = 0$: plastic or perfectly inelastic

$0 < e < 1$: inelastic.

Definition:



$$e = \frac{-(v_A - v_B)_f}{(v_A - v_B)}.$$

$$e = \frac{\text{speed of separation after collision}}{\text{speed of approach before collision}}$$

Example. Ball A hits ball B, while both go in a straight line. $m_A = 0.2$ kg, $m_B = 0.8$ kg., $V_{Ai} = 1$ m/s, $V_{Bi} = -1$ m/s.

Determine the post-collision velocities.

for 3 cases: a) $e = 1$, b) $e = 0.5$, c) $e = 0$.

Same method for all three cases.

2 unknowns V_{Af}, V_{Bf}. Need 2 equations.

(1) $m_A V_{Ai} + m_B V_{Bi} = m_A V_{Af} + m_B V_{Bf}$.

$$e = \frac{-(V_{Af} - V_{Bf})}{V_{Ai} - V_{Bi}}.$$

(2) $\Rightarrow (V_{Ai} - V_{Bi})\, e = -(V_{Af} - V_{Bf})$.

Solve the 2 equations in 2 unknowns.

DONE.

ANGULAR MOMENTUM

Consider a particle P moving with velocity $\bar{v}_P$ (with respect to some inertial frame).

Given some other point Q,

the angular momentum $\bar{H}_{P/Q}$ of particle P about point Q

is given by :

vector cross product

angular momentum

$$\boxed{\bar{H}_{P/Q} = \bar{r}_{P/Q} \times \underbrace{m\bar{v}_P}}$$

velocity need not be with respect to Q.

linear momentum

position vector from Q to P, $\overline{QP} = \bar{r}_{P/Q}$.

(Q need not be fixed in inertial frame)

(Sometimes called "moment of momentum").

What are the properties of this quantity?

Let us start again from "$F = ma$"

$$\Sigma \bar{F} = m\bar{a}_P = m\frac{d\bar{v}_P}{dt}.$$

$$\Rightarrow \quad \bar{r}_{P/Q} \times \Sigma \bar{F} = \bar{r}_{P/Q} \times m\frac{d\bar{v}_P}{dt}. \qquad (1)$$

LHS of (1) $= \bar{r}_{P/Q} \times \Sigma \bar{F}$ = Moment of all forces about Q. $= \Sigma \bar{M}_{/Q}$ (2)

= total moment about Q

RHS of (1) $= \bar{r}_{P/Q} \times m\dfrac{d\bar{v}_P}{dt} = \dfrac{d}{dt}\left(\bar{r}_{P/Q} \times m\bar{v}_P\right) = \dfrac{d}{dt}\bar{H}_{P/Q}$. (3)

why? ← if Q is fixed

Why? This is because $\dfrac{d}{dt}\left(\bar{r}_{P/Q} \times m\bar{v}_P\right) = \dfrac{d\bar{r}_{P/Q}}{dt} \times m\bar{v}_P + \bar{r}_{P/Q} \times m\dfrac{d\bar{v}_P}{dt}$.

$$= \bar{v}_{P/Q} \times m\bar{v}_P + \bar{r}_{P/Q} \times m\frac{d\bar{v}_P}{dt}$$

$$= \bar{0} + \bar{r}_{P/Q} \times m\frac{d\bar{v}_P}{dt}.$$

Putting (2) and (3) together, we have

(4) $$\boxed{\sum \bar{M}_{/Q} = \frac{d\bar{H}_{P/Q}}{dt}}$$ if Q is fixed.

Sum of all moments about Q] = rate of change of angular momentum

analogous to $\Sigma\bar{F} = m\bar{a}$

$$= \frac{d(m\bar{v})}{dt}$$

= rate of change of linear momentum.

Angular impulse momentum relation

$$\Sigma \bar{M}_{/Q} = \frac{d \bar{H}_{P/Q}}{dt}$$

Integrate both sides

$$\underbrace{\int_0^t \Sigma \bar{M}_{/Q}\, dt}_{\text{called "angular impulse"}} = \int_0^t \frac{d}{dt} \bar{H}_{P/Q} = \bar{H}_{P/Q} \Big|_0^t$$

$$= \underbrace{\bar{H}_{P/Q\,\text{final}} - \bar{H}_{P/Q\,\text{initial}}}_{\text{"change in angular momentum"}} .$$

$$\boxed{\int \Sigma \bar{M}_Q \, dt = \bar{H}_{P/Q\,\text{final}} - \bar{H}_{P/Q\,\text{initial}}} .$$

$$\boxed{\text{angular impulse integral} = \text{change in angular momentum}}$$

analogous to $\int \Sigma F\, dt$ = change in linear momentum.

When is angular momentum conserved?

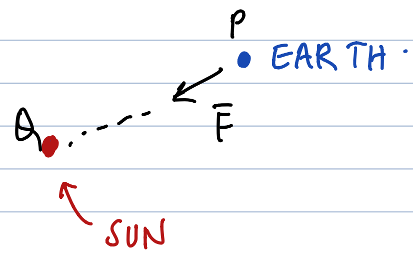
In general, if $\int_0^t \Sigma \hat{M}_{/Q}\, dt = \bar{0}$

then angular momentum is conserved.

- This can happen if $\Sigma \bar{M}_{/Q} = \bar{0}$. No net moments about point Q.

- Even more special case: "central forces"

Example.



- Ignore force due to other planets.
- Assume sun fixed.

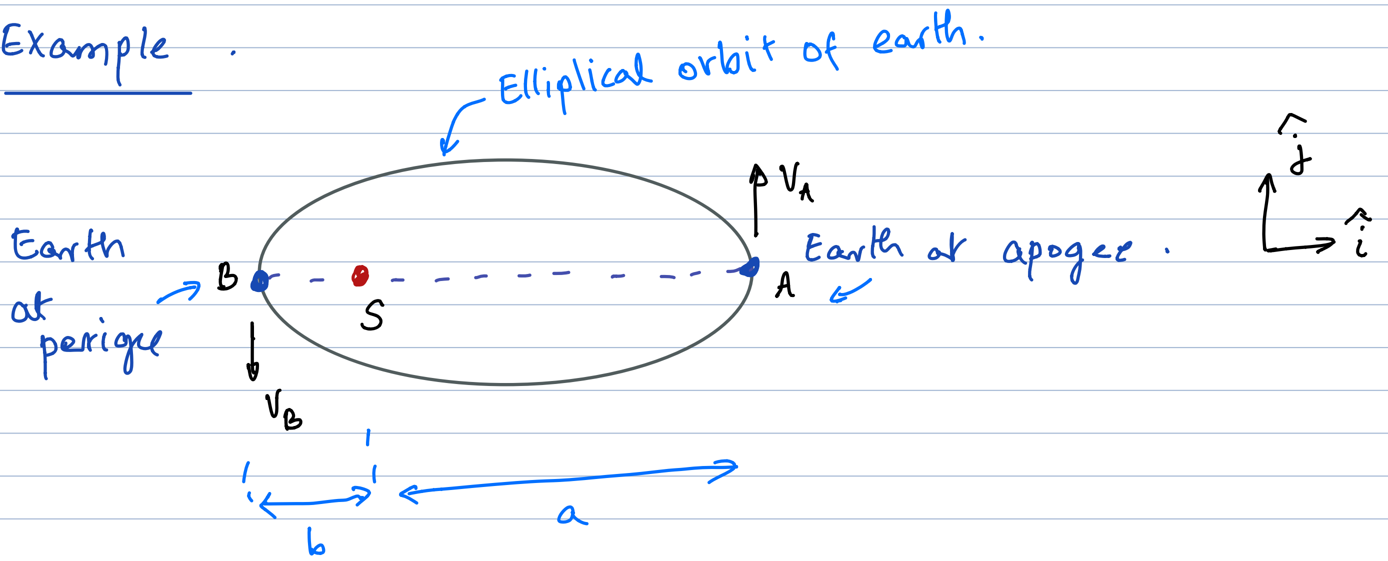
approximations.

- then $\Sigma \bar{M}_{/Q}$ = moment of forces on earth about the sun (Q)

$= \bar{r}_{P/Q} \times \bar{F} = \bar{0}$ because these are parallel vectors.

$\Rightarrow \bar{H}_{P/Q} = \bar{H}_{Earth/Sun}$ = conserved!

Example.



angular momentum conserved,

$$\bar{H}_{A/S} = \bar{r}_{A/S} \times m\bar{V}_A \quad \text{(angular momentum at A)}$$

$$= a\hat{i} \times m V_A \hat{j} = maV_A \hat{k}.$$

because $\hat{i} \times \hat{j} = \hat{k}$.

$$\bar{H}_{B/S} = \text{angular momentum at B}$$

$$= \bar{r}_{B/S} \times m\bar{V}_B = -b\hat{i} \times \left[m\left(-V_B\hat{j}\right)\right]$$

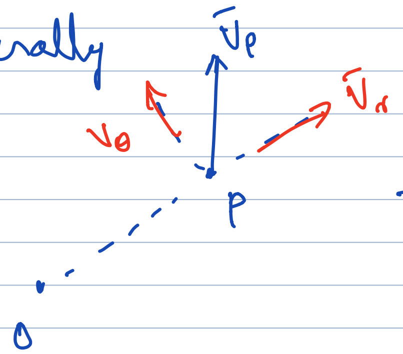
$$= mbV_B \hat{k}.$$

By angular momentum balance.

$$\cancel{m}aV_A \cancel{\hat{k}} = \cancel{m}bV_B \cancel{\hat{k}}$$

$$\boxed{aV_A = bV_B}$$

More generally



$$\bar{v}_P = v_r \hat{e}_r + v_\theta \hat{e}_\theta .$$
$$= \dot{r} \hat{e}_r + r\dot{\theta} \hat{e}_\theta .$$

$$\left.\begin{array}{c}\text{angular momentum} \\ \text{of P about O}\end{array}\right\} = \bar{r}_{P/O} \times m\bar{v}_P$$
$$= m r v_\theta \hat{k} = m r^2 \dot{\theta} \hat{k} .$$

(because $\bar{r}_{P/O} = r\hat{e}_r$ and $\hat{e}_r \times \hat{e}_r = \bar{0}$
$\hat{e}_r \times \hat{e}_\theta = \hat{k}$)

So, angular momentum balance $= r^2 \dot{\theta} =$ const.

or $r_1^2 \dot{\theta}_1 = r_2^2 \dot{\theta}_2$

"Kepler's second law".

All "dynamics equations" derived from "$\Sigma F = m\bar{a}$"

are ordinary differential equations (ODEs).

2^{nd} order ODEs.

2^{nd} derivative $\Rightarrow$ 2^{nd} order ODE

$$m\ddot{x} = \Sigma F$$

How to solve such ODEs?

(1) → analytical methods : closed form solutions or "pencil & paper solutions

↳ usually applies to simple problems.

eg. one object, one degree of freedom

(2) numerical methods : using a computer.

1000 s of books & methods.

- Euler's method - simplest method. ← we will do this

- Runge-kutta, more advanced. ← you may see this in your numerical methods class.

Euler's method.

Let's first do first order ODEs.

$$\frac{dx}{dt} = v(t)$$

find $x(t)$, given $v(t)$, $x(0)$.

$$dx = v(t)\,dt$$

$$\boxed{\Delta x = v(t)\,\Delta t\,.}$$

initial $x(0)$. say pick small $\Delta t = 0.0001$ s.

$$x(0.0001) = x(0) + v(0)\,\underbrace{\Delta t}_{0.0001.}$$

stepping forward in time.

$$x(0.0002) = x(0.0001) + v(0.0001)\cdot\Delta t$$

$$\vdots$$

in increments of $\Delta t = 0.0001$ s.

to get $x(t)$

actual time

$$x(\underbrace{i\,\Delta t}) = x\big((i-1)\Delta t\big) + v\big((i-1)\Delta t\big)\cdot\Delta t.$$

to simplify

$$\boxed{xList(i) = xList(i-1) + vList(i-1)\,\Delta t\,.}$$

Euler's method

$xList(i) = x(t)$ where $t = i(\Delta t)$.

for first order ODEs.

Second order ODEs

$$m\ddot{x} = F(t)$$

$$ma = F(t).$$

$$a(t) = F(t)/m.$$

use both equations to update v using a and update x using v.

$$\frac{dv}{dt} = a \Rightarrow \Delta v = a\,\Delta t$$

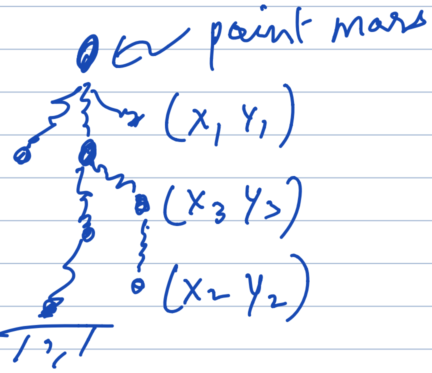
$$\frac{dx}{dt} = v \Rightarrow \Delta x = v\,\Delta t.$$

```
v(1) = x0;  v(1) = v0;

for i = 1 : numpoints          compute a(i-1) given.
    v(i) = v(i-1) + a(i-1) Δt.
    x(i) = x(i-1) + v(i-1) Δt)

end
```

Coupled system of many objects



$\ddot{x}_1 =$ —— $f(x_1\, y_1,\ x_2\, y_2 \ldots$ $v_{x_1}\, v_{x_2} \ldots$

$\ddot{y}_1 =$ —— complex function

$\ddot{x}_2 =$ —— of $x_i\, y_i,\ v_{x_i}\, v_{y_i}$

$\vdots$

for $i = 1:$ numpoints

$$v_{x_1}(i) = v_{x_1}(i-1) + a_{x_1}(i-1)\,\Delta t$$

$$x_1(i) = x_1(i-1) + v_{x_1}(i-1)\,\Delta t$$

v_{y_1} . . .

y_1

$\vdots$

end

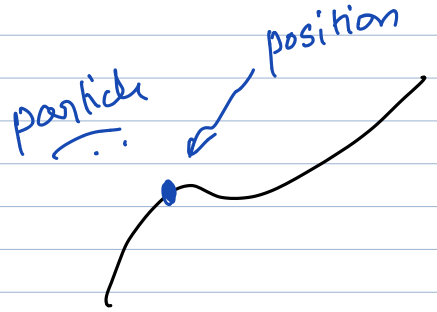
Thus for a more complex system, we just update the position & velocity of each & every object in each direction (could be linear or angular position / velocity).

Thus same method applies however complex the system.

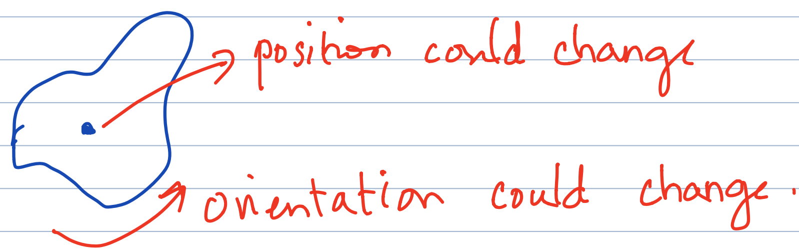
Need Δt to be small for the approximation to be good. See MATLAB.

So far we've looked at particles:

- kinematics (geometry + calculus)
- kinetics ("$F = ma$". relation between forces & motion).



Rigid body : Simplest object for which both position & orientation matter.



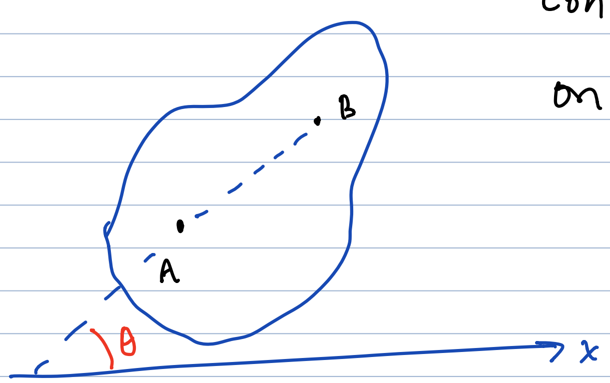
Deformable body : can change position, orientation, shape

Rigid body : does not change shape.

Angular velocity for a rigid body (2D).

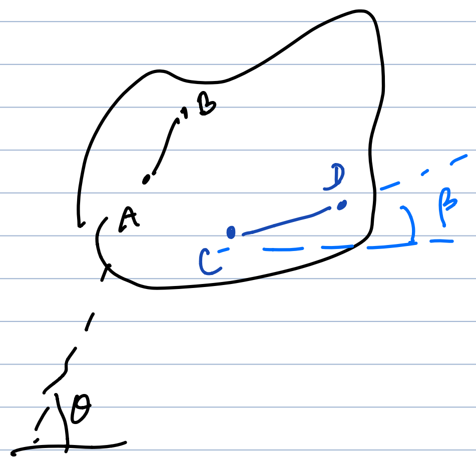
Consider any 2 points A and B on the rigid body.

θ = angle made by line AB with some fixed axis, in the plane, say x-axis.



$$\frac{d\theta}{dt} = \text{angular velocity (scalar) of the rigid body.} = \omega$$

angular velocity vector $\bar{\omega} = \omega \hat{k}$



$\theta - \beta$ = constant because rigid body

$$\Rightarrow \frac{d\theta}{dt} - \frac{d\beta}{dt} = 0.$$

$$\dot{\theta} = \dot{\beta}$$

$$\bar{\omega} = \dot{\theta}\hat{k} = \dot{\beta}\hat{k}.$$

So, angular velocity $\bar{\omega}$ is a property of the object as a whole. (does not depend on choice of A, B, C, D).

Also, we do not have to say, "angular velocity about a point".

Angular acceleration $\bar{\alpha}$ (vector)

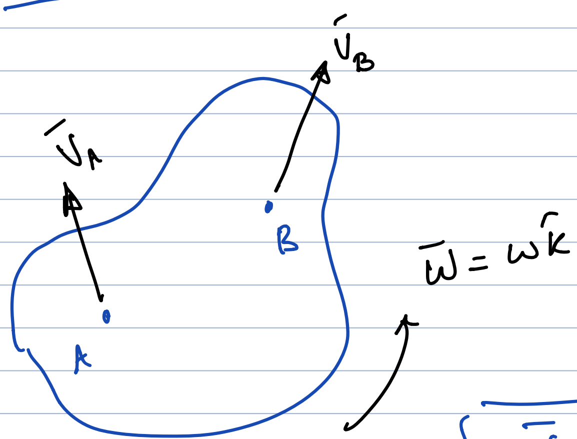
α (scalar component)

$$\boxed{\bar{\alpha} = \frac{d\bar{\omega}}{dt}} = \frac{d\omega}{dt}\hat{k} = \alpha\hat{k}.$$

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}}$$

Property of the rigid body's motion as a whole.

Two big relations in rigid body kinematics



(1) Relates $\bar{V}_A$ and $\bar{V}_B$ for any 2 A and B on the same rigid body.

Arrh.

$$\boxed{\bar{V}_A = \bar{V}_B + \bar{\omega} \times \bar{r}_{A/B}} \quad - (1)$$

$$\equiv \quad \bar{V}_A - \bar{V}_B = \bar{\omega} \times (\bar{r}_A - \bar{r}_B)$$

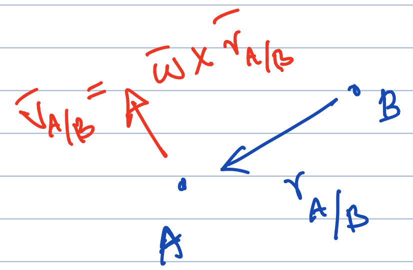
$$\boxed{\bar{V}_{A/B} = \bar{\omega} \times \bar{r}_{A/B}}$$

vector cross product

Because A and B are on a rigid body, the only relative motion between A and B can be a rotation of A about the other.

$$\bar{V}_{A/B} = \bar{\omega} \times \bar{r}_{A/B}$$

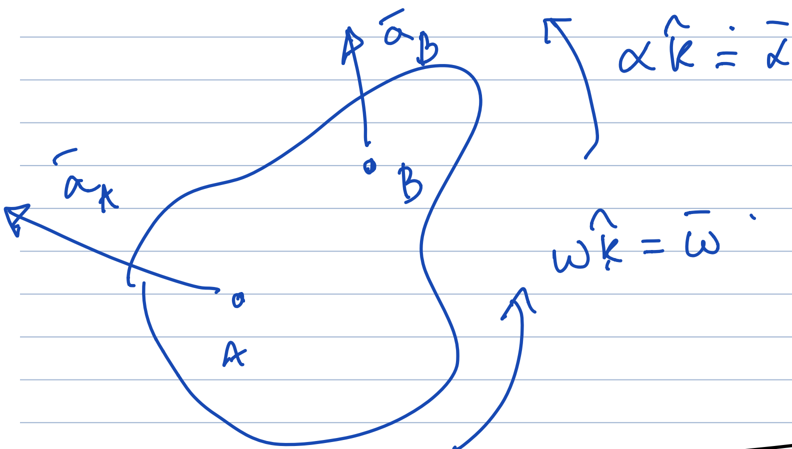
$\bar{\omega} = \omega\hat{k}$ - perpendicular to plane



$\bar{V}_{A/B}$ perpendicular to $\hat{k}$ & $\bar{r}_{A/B}$. as if A was rotating about B.

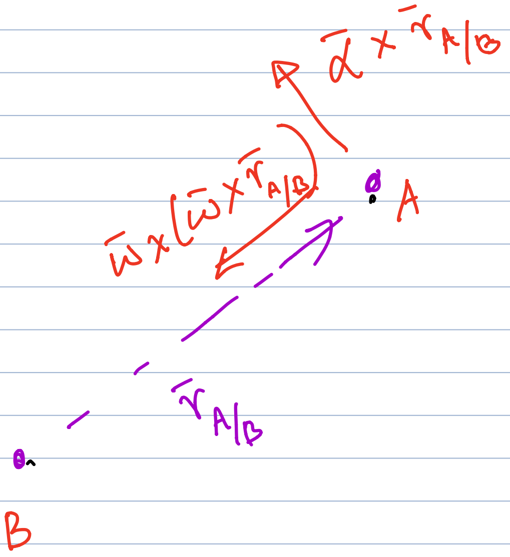
$(\bar{c} = \bar{a} \times \bar{b} \Rightarrow \bar{c} \perp^{ar}$ to $\bar{a}$ and $\bar{b})$.

Relation 2 : accelerations of A and B



$$\boxed{\bar{a}_A = \bar{a}_B + \bar{\alpha} \times \bar{r}_{A/B} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{A/B})}.$$

cross products.



$\bar{\omega} \times (\bar{\omega} \times \bar{r}_{A/B})$ -

centripetal acceleration of A wrt B

with respect to.

<u>Example</u>. Given some rigid body. A & B on it.

Given $\bar{V}_A = 3\hat{i} + 4\hat{j}$, $\bar{\omega} = 5\hat{k}$.

$A(0,5)$, $B(3,7)$ in cartesian coordinates.

Determine $\bar{V}_B$.

$$\bar{V}_B = \bar{V}_A + \bar{\omega} \times \bar{r}_{B/A}$$

$$\bar{r}_{B/A} = \bar{r}_B - \bar{r}_A = (3\hat{i} + 7\hat{j}) - (0\hat{i} + 5\hat{j}).$$

$$\bar{r}_{B/A} = 3\hat{i} + 2\hat{j}.$$

$$\bar{V}_B = (3\hat{i} + 4\hat{j}) + (5\hat{k}) \times (3\hat{i} + 2\hat{j}).$$

$$= 3\hat{i} + 4\hat{j} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 5 \\ 3 & 2 & 0 \end{vmatrix} \leftarrow \text{determinant}$$

$$= 3\hat{i} + 4\hat{j} + \hat{i}(0 - 2\times5) - \hat{j}(0 - 3\times5)$$
$$+ \hat{k}(0-0)$$

$$= 3\hat{i} + 4\hat{j} - 10\hat{i} + 15\hat{j} + 0\hat{k}.$$

$$\boxed{\bar{V}_B = -7\hat{i} + 19\hat{j}.}$$

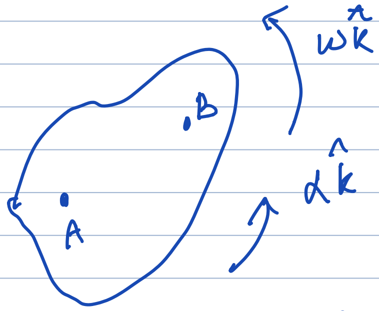
<u>Alternative</u> to <u>determinant</u>

$\hat{i} \times \hat{j} = \hat{k}$	$\hat{j} \times \hat{i} = -\hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$	$\hat{k} \times \hat{j} = -\hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$	$\hat{i} \times \hat{k} = -\hat{j}$

$$\begin{tikzcd} \hat{k} \arrow[d, bend right] & \hat{j} \arrow[l] \\ \hat{i} \arrow[ur] & \end{tikzcd}$$

$$5\hat{k} \times (3\hat{i} + 2\hat{j})$$
$$= 15\,\hat{k}\times\hat{i} + 10\,\hat{k}\times\hat{j}$$
$$= 15\hat{j} + 10(-\hat{i})$$

RIGID BODY KINEMATICS (CONTINUED).



$\bar{r}_{A/B}$

① $\bar{v}_A = \bar{v}_B + \bar{\omega} \times \underbrace{(\bar{r}_A - \bar{r}_B)}_{\bar{r}_{A/B}}$.

② $\bar{a}_A = \bar{a}_B + \bar{\alpha} \times (\bar{r}_A - \bar{r}_B) + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{A/B})$

② $\equiv \bar{a}_A = \bar{a}_B + \bar{\alpha} \times \bar{r}_{A/B} - \omega^2 \bar{r}_{A/B}$

easier to use.

scalar ω (magnitude of ω)

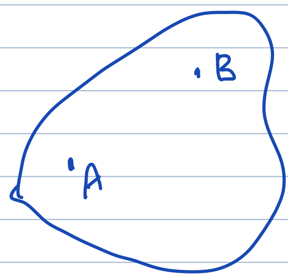
$\bar{\omega} = \omega \hat{k}$

vector angular velocity

Scalar component

Example

Given $A(2,3)$, $B(4,6)$ in Cartesian (m)



$\bar{\omega}$ 3 rad/s anticlockwise.

$\bar{\alpha}$ 4 rad/s² clockwise

$\bar{a}_B = 3\hat{i} + \hat{j}$ m/s²

Q: $\bar{a}_A = ?$

Solution By convention, +ve α, ω is anticlockwise.

$\bar{\omega} = 3\hat{k}$ (rad/s)

$\bar{\alpha} = -4\hat{k}$ (rad/s²)

clockwise

$\bar{r}_A = 2\hat{i} + 3\hat{j}$

$\bar{r}_B = 4\hat{i} + 6\hat{j}$

$\bar{r}_{A/B} = (-2\hat{i} - 3\hat{j})$

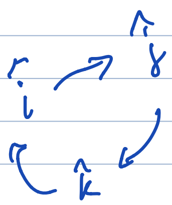
$$\bar{a}_A = \bar{a}_B + \bar{\alpha} \times \bar{r}_{A/B} - \omega^2 \bar{r}_{A/B}$$

$$= (3\hat{i} + \hat{j}) + (-4\hat{k}) \times (-2\hat{i} - 3\hat{j}) - 3^2(-2\hat{i} - 3\hat{j}).$$

$$= 3\hat{i} + \hat{j} + (4 \times 2)(\hat{k} \times \hat{i}) + (4 \times 3)(\hat{k} \times \hat{j}) + 18\hat{i} + 27\hat{j}$$

$$= 3\hat{i} + \hat{j} + 8(\hat{j}) + 12(-\hat{i}) + 18\hat{i} + 27\hat{j}$$

$$= \hat{i}(3 - 12 + 18) + \hat{j}(1 + 8 + 27)$$



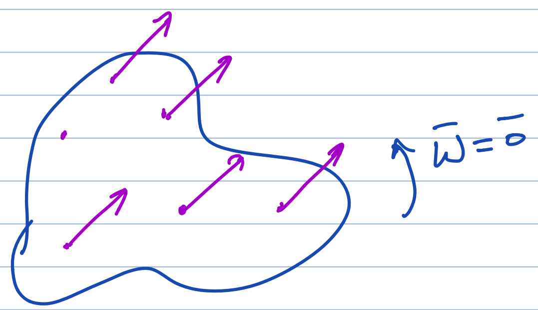
$\hat{i} \times \hat{j} = \hat{k}$ | $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$ | $\hat{k} \times \hat{j} = -\hat{i}$

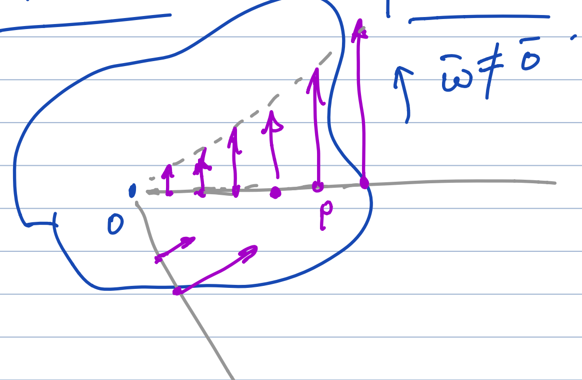
$\hat{k} \times \hat{i} = \hat{j}$ | $\hat{i} \times \hat{k} = -\hat{j}$

Pure translation

all points have same velocity.



(Pure) Rotation about point O. $\bar{v}_O = \bar{0}$ (O is not moving).



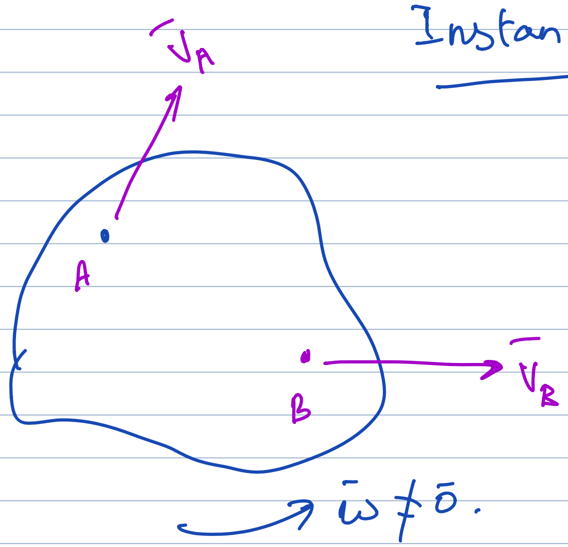
Velocity of P's magnitude scales linearly with distance from O & perpendicular to $\bar{r}_{P/O}$

Follows from $\bar{v}_p = \cancel{\bar{v}_o}^{\,0} + \bar{\omega} \times \bar{r}_{p/o}$

$$= \underbrace{\bar{\omega} \times \bar{r}_{p/o}}$$

perpendicular to $\bar{r}_{p/o}$

and is bigger if $|\bar{r}_{p/o}|$ is bigger.



Instantaneous center of rotation

If $\bar{\omega} \neq \bar{0}$, then

there is some point Q

on or off the rigid

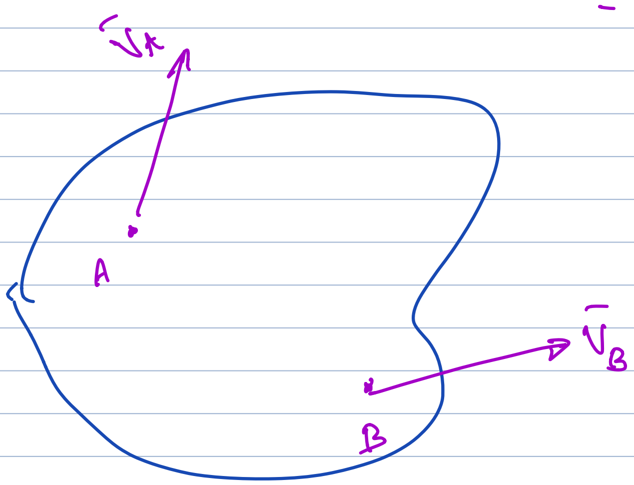
body, for which

$\bar{v}_Q = \bar{0}$.

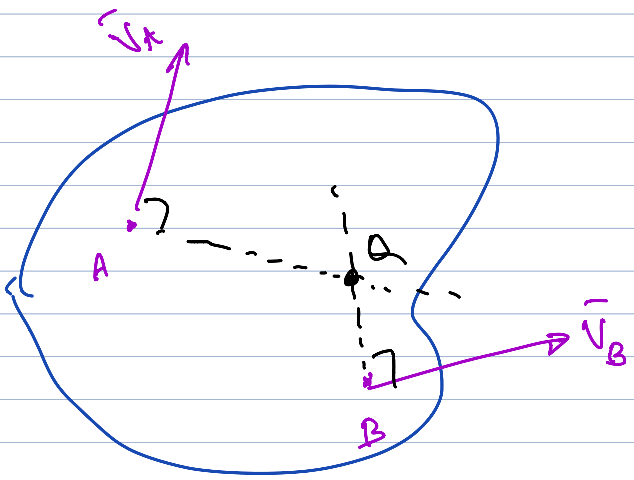
It is as if the whole rigid body rotates about Q, at the moment.

Q = instantaneous center of rotation.

How to find it?



- Given $\bar{v}$ of 2 points (A & B) draw perpendiculars to those velocities $\bar{v}_A$ and $\bar{v}_B$ starting from A and B.



- Intersection of these perpendiculars ≡ instantaneous center of rotation.

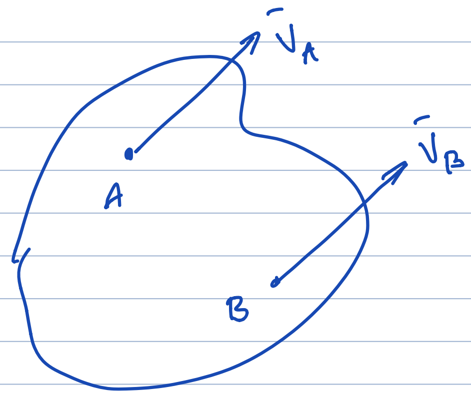
perpendicular to $\bar{v}_c$ for any c will also pass through Q.

Q can be outside the body.

- Two special cases:

If A and B have velocities that are parallel.

Case 1

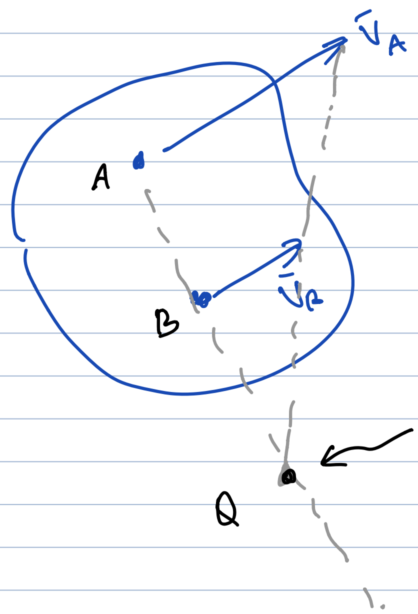


Say $\bar{V}_A = \bar{V}_B$ is known

$\Rightarrow \bar{\omega} = \bar{0}$ or

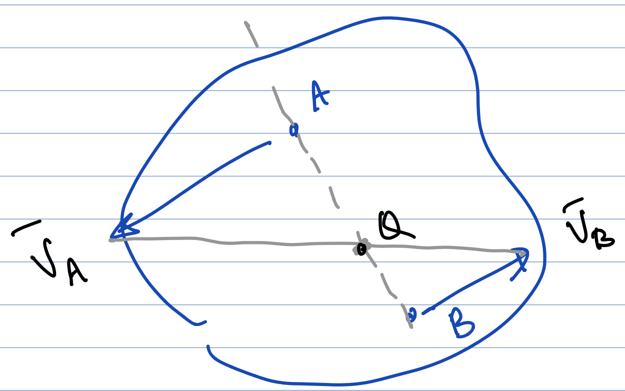
pure translation

Case 2 $\bar{V}_A$ and $\bar{V}_B$ not equal but parallel.

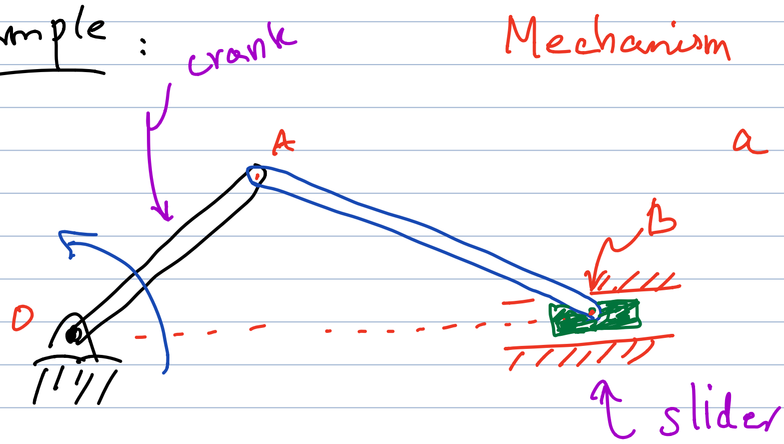


- Connect & extend AB.
- Connect & extend heads of $\bar{V}_A$ & $\bar{V}_B$.
  - intersection

instantaneous center of rotation



Example :



Mechanism OAB is called a "slider-crank mechanism"

a few tens of billions in the world.

all IC engine cars, etc.

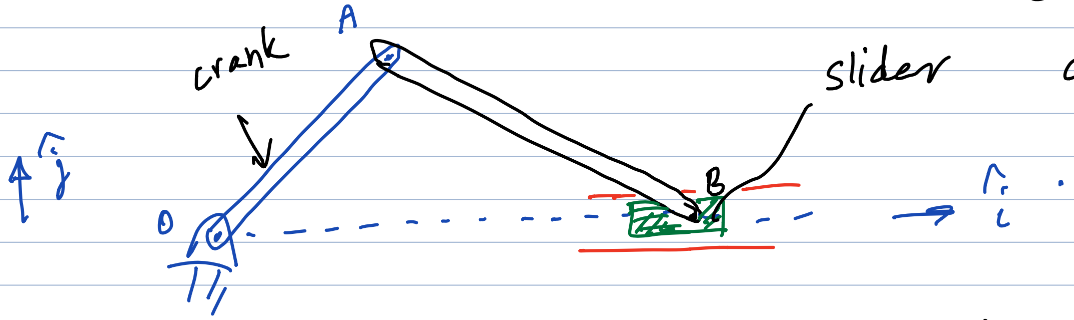
Kinds of problems : given $\bar{\omega}_{OA}$ and/or $\bar{\alpha}_{OA}$

find $\bar{v}_B$ and/or $\bar{a}_B$ and/or

$\bar{\omega}_{AB}$ and/or $\bar{\alpha}_{AB}$

(Some of these qtys given & asked other of these qtys).

Example :



OA and AB are rigid bodies.

At the moment shown, $O(0,0)$, $A(1,1)$, $B(3,0)$

a) Given, $\bar{\omega}_{OA}$ is 2 rad/s anticlockwise.

Determine $\bar{v}_A$, $\bar{\omega}_{AB}$ and $\bar{v}_B$.

Solution: $\bar{v}_A = \bar{v}_O + \bar{\omega}_{OA} \times \bar{r}_{A/O}$ because O & A are on the same rigid body.

$\bar{v}_O = \bar{0}$

$\bar{\omega}_{OA} = +2\hat{k}$, $\bar{r}_{A/O} = (\hat{i} + \hat{j}) - (\bar{0})$

$= \hat{i} + \hat{j}$ (because $A(1,1)$).

$$\bar{v}_A = \bar{0} + 2\hat{k} \times (\hat{i} + \hat{j}) = 2(\hat{k} \times \hat{i}) + 2(\hat{k} \times \hat{j})$$

$$= 2\hat{j} + 2(-\hat{i}) = \boxed{-2\hat{i} + 2\hat{j}}. \quad \text{ANS}$$

$\bar{\omega}_{AB} = \omega_{AB}\hat{k}$, where scalar ω_{AB} is unknown.

$\bar{v}_B = v_B \hat{i}$, because only in $\hat{i}$ direction.

& v_B scalar is unknown.

2 scalar unknowns : ω_{AB} and v_B. Need 2 scalar equations.

Because A & B are on same rigid body (AB).

$$\bar{V}_A = \bar{V}_B + \bar{\omega}_{AB} \times \bar{r}_{A/B}, \quad \text{where } \bar{r}_{A/B} = [\hat{i}+\hat{j}] - (3\hat{i}+\bar{0}).$$
$$= -2\hat{i}+\hat{j}$$

$$-2\hat{i}+2\hat{j} = V_B\hat{i} + (\omega_{AB}\hat{k}) \times (-2\hat{i}+\hat{j}).$$

$$-2\hat{i}+2\hat{j} = V_B\hat{i} - 2\omega_{AB}(\hat{k}\times\hat{i}) + \omega_{AB}(\hat{k}\times\hat{j})$$

$$-2\hat{i}+2\hat{j} = V_B\hat{i} - 2\omega_{AB}\hat{j} + \omega_{AB}(-\hat{i})$$

Equating $\hat{i}$ terms:

$$-2 = V_B - \omega_{AB} \quad (1)$$

$\hat{j}$:
$$2 = -2\omega_{AB} \quad (2)$$

$$(2) \Rightarrow \omega_{AB} = \frac{2}{-2} = \boxed{-1 \text{ rad/s}}$$ ANS

$$\boxed{\bar{\omega}_{AB} = -1\hat{k} \text{ rad/s}}$$ ANS

Plug in (1):

$$-2 = V_B - (-1) = V_B + 1$$

$$\boxed{V_B = -3 \text{ m/s}}$$

$$\boxed{\bar{V}_B = -3\hat{i} \text{ m/s}}$$ ANS

(b) In addition to info in part-a, you're told that $\bar{a}_B = 2\hat{i}\ m/s^2$. Determine $\bar{\alpha}_{OA}$.

Solution: $\bar{\alpha}_{AB} = \alpha_{AB}\hat{k}$, $\bar{\alpha}_{OA} = \underbrace{\alpha_{OA}}\hat{k}$.

need to find.

AB is a rigid body. so. use standard formula.

(3) $$\underbrace{\bar{a}_A}_{\text{unknown}} = \underbrace{\bar{a}_B}_{\text{given}} + \underbrace{\alpha_{AB}}_{\text{unknown}}\hat{k} \times \underbrace{\bar{r}_{A/B}}_{\text{known}} - \underbrace{\omega_{AB}^2}_{\substack{\text{known}\\ \text{from (a)}}} \underbrace{\bar{r}_{A/B}}_{\text{known}}$$

OA is a rigid body. so

(4) $$\bar{a}_A = \bar{a}_o + \alpha_{OA}\hat{k} \times \bar{r}_{A/O} - \omega_{OA}^2 \bar{r}_{A/O}$$

Equate $\bar{a}_A$ from (3) & (4) [or do (3) − (4)]

$$\underbrace{\bar{a}_B}_{\checkmark} + \underbrace{\alpha_{AB}}_{?}\hat{k} \times \underbrace{\bar{r}_{A/B}}_{\checkmark} \underbrace{- \omega_{AB}^2 \bar{r}_{A/B}}_{\checkmark} = \underbrace{\bar{a}_o}_{\checkmark = 0} + \underbrace{\alpha_{OA}}_{?}\hat{k} \times \underbrace{\bar{r}_{A/O}}_{\checkmark} \underbrace{- \omega_{OA}^2 \bar{r}_{A/O}}_{\checkmark}$$

1 vector equation with 2 scalar unknowns α_{AB} & α_{OA}.

Solve.

$$2\hat{i} + \alpha_{AB}\hat{k} \times (-2\hat{i} + \hat{j}) - (-1)^2(-2\hat{i} + \hat{j})$$

$$= \bar{0} + \alpha_{OA}\hat{k} \times (\hat{i}+\hat{j}) - 2^2(\hat{i}+\hat{j})$$

So

$$2\hat{i} - 2\alpha_{AB}(\hat{k}\times\hat{i}) + \alpha_{AB}(\hat{k}\times\hat{j}) + 2\hat{i} - \hat{j}$$

$$= \alpha_{OA}(\hat{k}\times\hat{i}) + \alpha_{OA}(\hat{k}\times\hat{j}) - 4\hat{i} - 4\hat{j}$$

$$\Rightarrow 2\hat{i} - 2\alpha_{AB}\hat{j} - \alpha_{AB}\hat{i} + 2\hat{i} - \hat{j}$$

$$= \alpha_{OA}\hat{j} - \alpha_{OA}\hat{i} - 4\hat{i} - 4\hat{j}$$

$\underline{\underline{\hat{i}}}$: $\quad 2 - \alpha_{AB} + 2 = -\alpha_{OA} - 4 \quad (5)$

$\underline{\underline{\hat{j}}}$: $\quad -2\alpha_{AB} - 1 = \alpha_{OA} - 4 \quad (6)$

Solve (5) & (6) for α_{OA} & α_{AB}.

from (5) $\quad \alpha_{AB} = 2+2+\alpha_{OA}+4 \quad (7)$

$$= 8 + \alpha_{OA}.$$

(7) $\rightarrow$ (6) :

$$-2(8+\alpha_{OA}) - 1 = \alpha_{OA} - 4.$$

$$-16 - 2\alpha_{OA} - 1 = \alpha_{OA} - 4$$

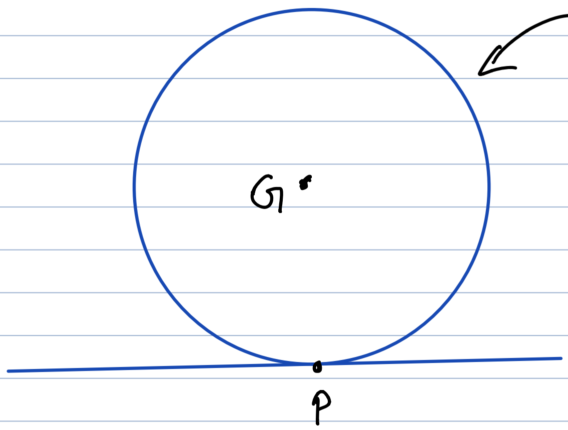
$$\Rightarrow 3\alpha_{OA} = -17 + 4 = -13$$

$$\boxed{\alpha_{OA} = -13/3 \quad rad/s^2}$$

$$\boxed{\bar{\alpha}_{OA} = -\frac{13}{3}\hat{k} \quad rad/s^2}$$

Next time: another example, with "four bar linkage".

Rolling without slip



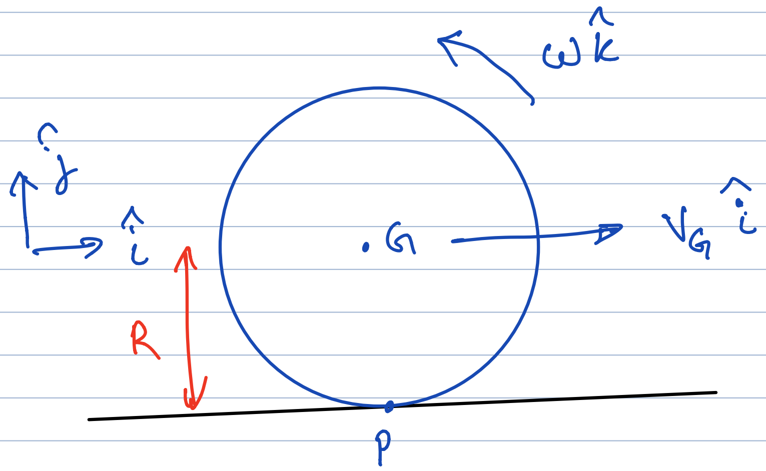
Wheel rolling without slip on a flat surface means that at every point in time, the

velocity of point P on the wheel $= \bar{0}$.

if $\bar{v}_P \neq \bar{0}$, then $\bar{v}_P$ is the slip velocity of P with respect to ground.

Here, P is the point on wheel currently making contact with ground. NOT a fixed point on the wheel.

Condition for rolling without slip



Rolling without slip

$$\Rightarrow \bar{V}_p = \bar{0}$$

$$\bar{V}_p = \bar{V}_G + \omega\hat{k} \times (\bar{r}_{P/G})$$

$$\bar{0} = V_G\hat{i} + \omega\hat{k} \times (-R\hat{j})$$

$$= V_G\hat{i} - R\omega(-\hat{i})$$

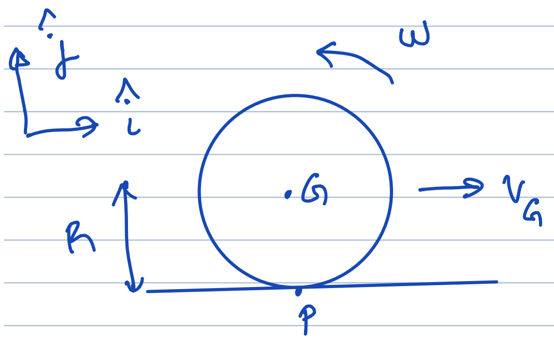
$$\boxed{V_G + R\omega = 0}$$

$$\boxed{V_G = -R\omega}$$

Condition for no slip

(relation between V_G & ω such that $\bar{V}_p = 0$

ROLLING WITHOUT SLIP.



P is the point on wheel that is currently in contact with ground.

G - center of wheel.

Rolling without slip $\Rightarrow$ $\bar{V}_P = \bar{0}$ (point in contact with ground has zero velocity).

slip $\equiv$ $\bar{V}_P$ (if not $\bar{0}$).

Relation between V_G and ω that ensures $\bar{V}_P = \bar{0}$

$$\bar{V}_P = \bar{0}$$

$$\bar{V}_P = \bar{V}_G + \bar{\omega} \times \bar{r}_{P/G}$$

$$= V_G \hat{i} + (\omega \hat{k}) \times (-R\hat{j})$$

$$= V_G \hat{i} - R\omega (\hat{k} \times \hat{j})$$

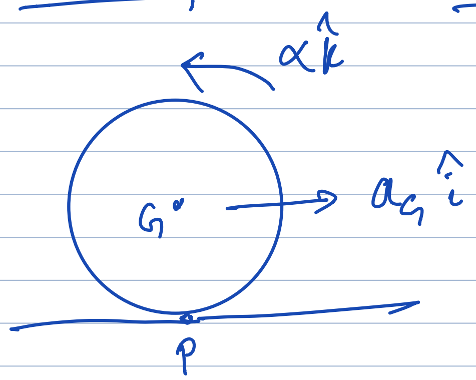
$$\bar{V}_P = V_G \hat{i} - R\omega(-\hat{i}) = (V_G + R\omega)\hat{i}.$$

$$\boxed{\bar{V}_P = (V_G + R\omega)\hat{i}}$$

If $\bar{v}_p = \bar{0}$ (rolling without slip)

then $\boxed{v_G = -R\omega}$

Relation for $\bar{a}_G$ and $\bar{\alpha}$



It turns out we cannot assume that $\bar{a}_p = \bar{0}$.
even though $\bar{v}_p = \bar{0}$.

Relation between a_G & α is obtained by differentiating relation between v_G & ω.

$$v_G = -R\omega$$

Differentiate $\Rightarrow$ $a_G = -R\dfrac{d\omega}{dt} = -R\alpha.$

$$\boxed{a_G = -R\alpha}$$

---

what is the acceleration $\bar{a}_p$ if not $\bar{0}$?

$$\bar{a}_p = \bar{a}_G + \bar{\alpha} \times \bar{r}_{P/G} - \omega^2 \bar{r}_{P/G}$$

$$= a_G\hat{i} + \alpha\hat{k} \times (-R\hat{j}) - \omega^2(-R\hat{j})$$

$$= a_G\hat{i} - R\alpha(\hat{k} \times \hat{j}) + \omega^2 R\hat{j}$$

$$= a_G \hat{i} - R\alpha(-\hat{i}) + R\omega^2 \hat{j})$$

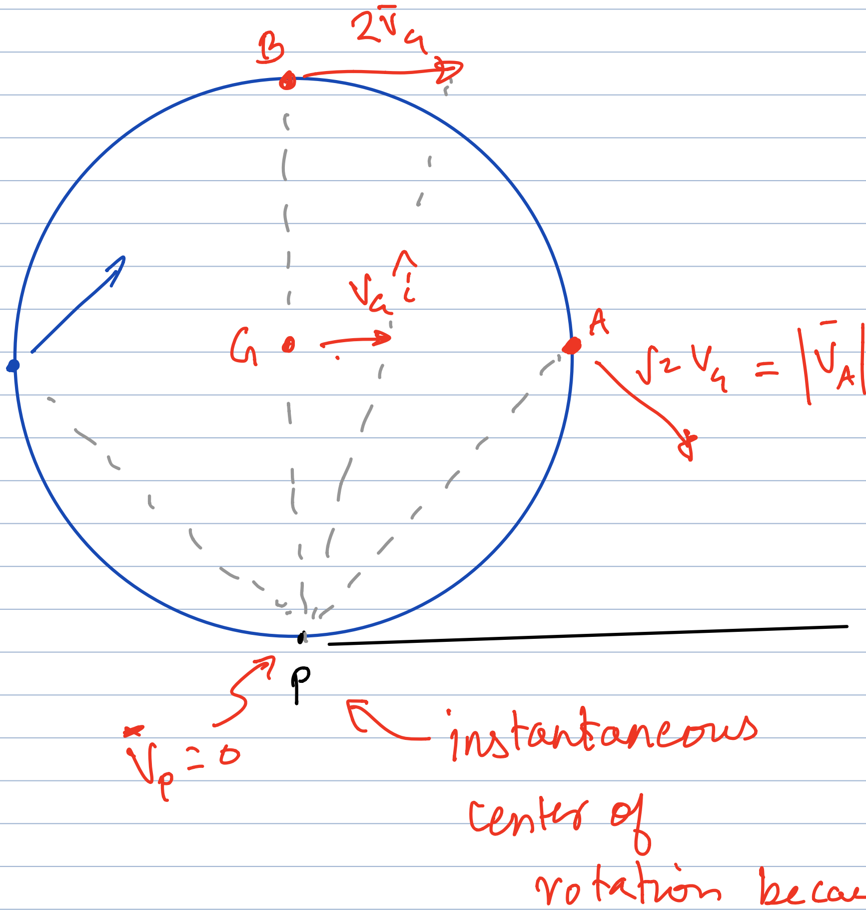
$$= \left(a_G + R\alpha\right)\hat{i} + R\omega^2 \hat{j}$$

$$= 0\hat{i} + R\omega^2 \hat{j}$$ because $a_G = -R\alpha$ for rolling without slip.

$$\boxed{\bar{a}_P = R\omega^2 \hat{j}}$$

↑ accelerates upward (related to centripetal acceleration).

↑ this is the acceleration that "pulls" the point P up off the ground.



$\bar{v}_P = \bar{0}$

$v_G = -R\omega.$

$\bar{v}_B = \bar{v}_P + \omega \hat{k} \times \bar{r}_{B/P}$

$= \bar{0} + \left(\frac{-v_G}{\cancel{R}}\right)\hat{k} \times (2\cancel{R}\hat{j})$

$= -v_G(\hat{k} \times \hat{j})\,2$

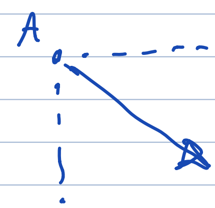
$= -2v_G(-\hat{i})$

$= 2v_G \hat{i}$

$$\bar{v}_A = \bar{v}_P + \omega\hat{k} \times \bar{r}_{A/P} = \bar{0} + \left(\frac{-v_G}{\cancel{R}}\right)\hat{k} \times (\cancel{R}\hat{i} + \cancel{R}\hat{j})$$

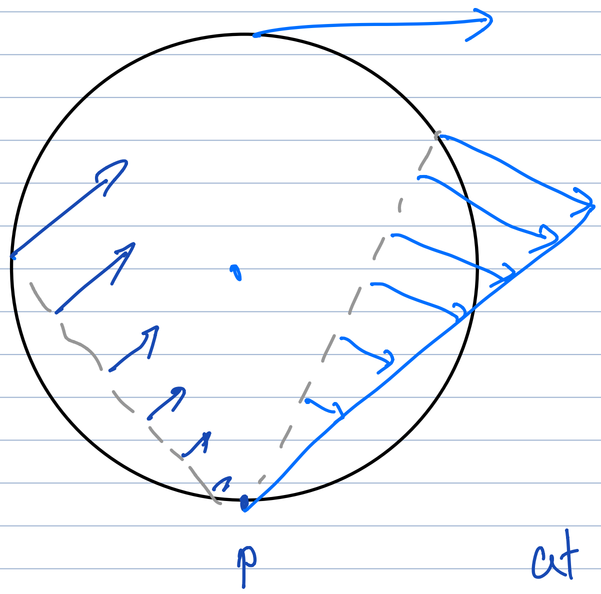
$$= -v_G(\hat{k}\times\hat{i}) - v_G(\hat{k}\times\hat{j})$$

$$= -v_G\hat{j} + v_G\hat{i}$$



$$|\bar{v}_A| = \sqrt{v_G^2 + v_G^2}$$

$$= \sqrt{2v_G^2} = v_G\sqrt{2}$$

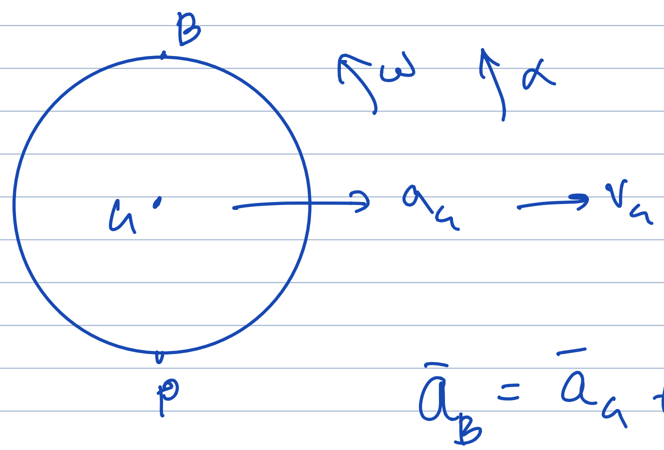


velocity of different points on the wheel can be understood by thinking that the wheel at the moment is rotating about P.

BUT, accelerations do not behave like pure rotations about P. ONLY velocities.

Just use the formula: $\bar{a}_Q = \bar{a}_P + \bar{\alpha} \times \bar{r}_{Q/P} - \omega^2 \bar{r}_{Q/P}$.

Ex: What is $\bar{a}_B$? (top-most point?).



$a_G + R\alpha = 0$
$v_G + R\omega = 0$] no slip.

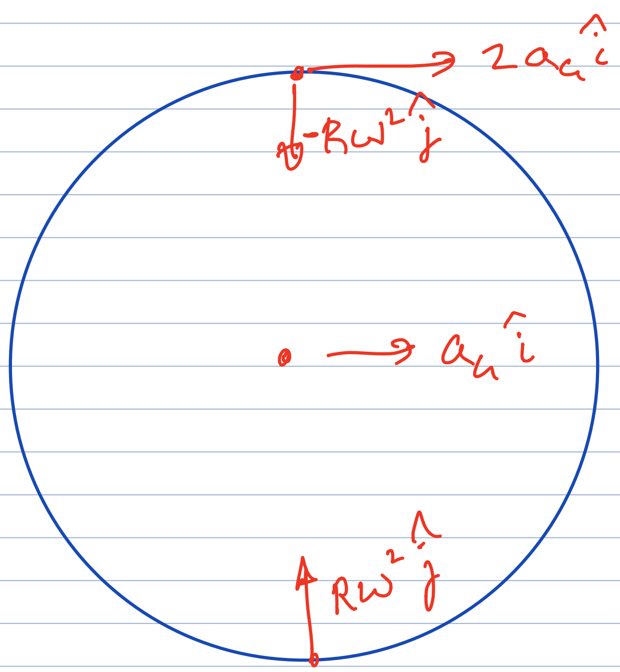
$$\bar{a}_B = \bar{a}_G + \alpha \hat{k} \times \bar{r}_{B/G} - \omega^2 \bar{r}_{B/G}$$

$$= a_G \hat{i} + \left(\frac{-a_G}{+R}\right) \hat{k} \times (R\hat{j}) - \omega^2 (R\hat{j})$$

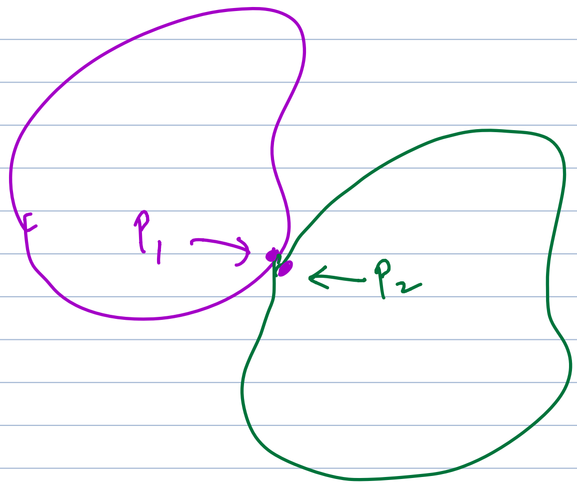
$$= a_G \hat{i} - \frac{a_G}{\cancel{R}} \cancel{R} (\hat{k} \times \hat{j}) - \omega^2 R \hat{j}$$

$$= a_G \hat{i} + a_G \hat{i} - \omega^2 R \hat{j}$$

$$= 2a_G \hat{i} - \omega^2 R \hat{j}$$



More general rolling without slip



2 objects contact each other at P_1 and P_2, respectively on the 2 objects.

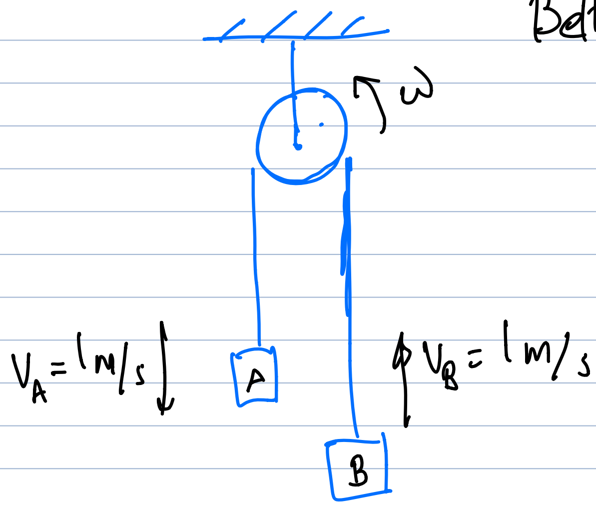
P_1 on object 1

P_2 on object 2.

Condition for rolling without slip = $\bar{v}_{P_1} = \bar{v}_{P_2}$.

(contact points have no relative velocity).

Example pulleys



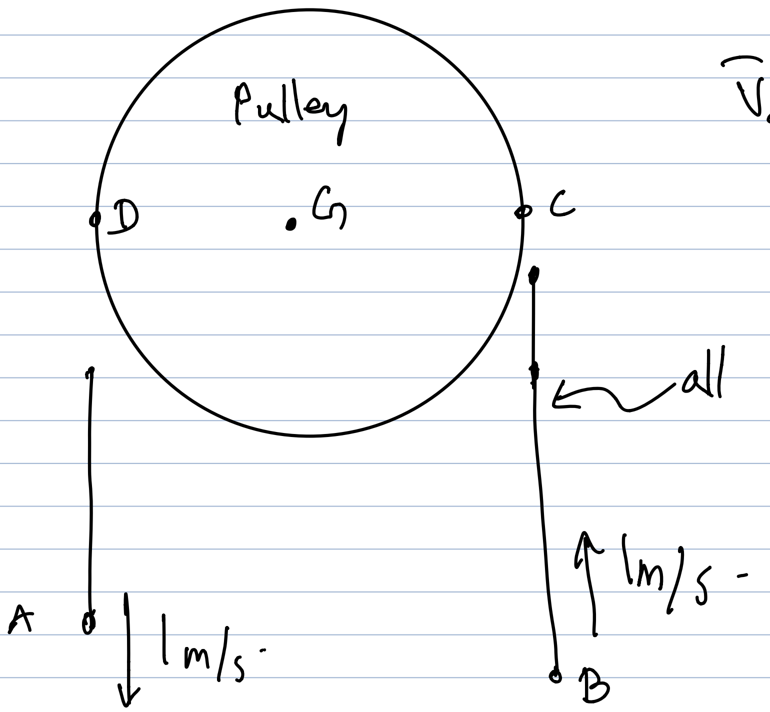
Belt has no slip on the pulley.

Given $V_A \downarrow$ and $V_B \uparrow = 1\,m/s$.

what is ω if

$R_{pulley} = 0.3\ m$.

Think of pulley rolling without slip on belts.



$\bar{V}_G = \bar{0}$ (G not moving).

all points on belt have same velocity until the point that touches C on pulley.

So for no slip.

we need

$\bar{V}_C = 1\hat{j}$. (C on pulley).

$V_D = -1\hat{j}$

for no slip with the belt on the left side (connected to A)..

$\bar{V}_G = \bar{0}$

$\bar{V}_C = \bar{V}_G + \omega\hat{k} \times \underbrace{(R\hat{i})}_{\bar{r}_{C/G}}$

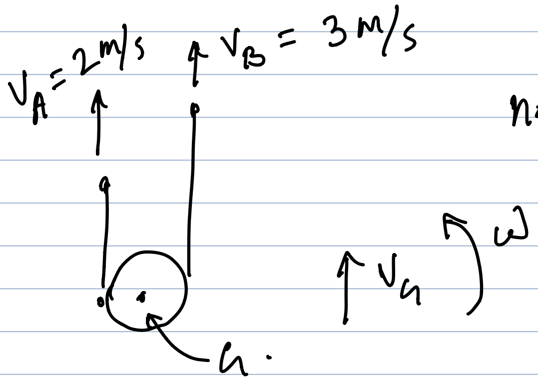
$1\hat{j} = \bar{0} + \omega\hat{k} \times (0.3\,\hat{i})$

$= 0.3\omega\,\hat{j}$

$0.3\omega = 1$

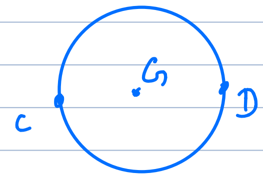
$\boxed{\omega = 1/0.3 \text{ rad/s}}$

Ex



no slip : determine V_G & ω

Hint



assume $\bar{V}_G = V_G\,\hat{j}$, $\bar{V}_C = \bar{V}_A = 2\,\hat{j}$

$\bar{V}_D = \bar{V}_B = 3\,\hat{j}$.

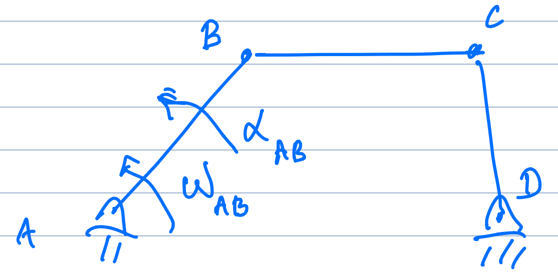
and then do : $\bar{V}_C = \bar{V}_G + \omega\hat{k} \times \bar{r}_{C/G}$

and $\bar{V}_D = \bar{V}_G + \omega\hat{k} \times \bar{r}_{D/G}$.

You will get 2 equations in 2 unknowns

V_G and ω. SOLVE.

Four-bar linkage (eg. Q4 on HW7)



ω_{AB}, α_{AB} given.

given positions of A, B, C, D. So we can compute

$\bar{r}_{A/B}$, $\bar{r}_{B/A}$, $\bar{r}_{C/B}$, $\bar{r}_{D/C}$.

Asked a) ω_{BC}, ω_{CD}

b) α_{BC}, α_{CD}.

For part a, do:

$$\bar{v}_B = \underbrace{\bar{v}_A}_{\bar{0}\ \checkmark} + \underbrace{\omega_{AB}}_{\checkmark}\hat{k} \times \underbrace{\bar{r}_{B/A}}_{\checkmark} \quad (1)$$

$$\underbrace{\bar{v}_C}_{?} = \underbrace{\bar{v}_B}_{\checkmark} + \underbrace{\omega_{BC}}_{?}\hat{k} \times \underbrace{\bar{r}_{C/B}}_{\checkmark} \quad (2)$$

$$\underbrace{\bar{v}_D}_{\checkmark \to \bar{0}} = \underbrace{\bar{v}_C}_{?} + \underbrace{\omega_{CD}}_{?}\hat{k} \times \underbrace{\bar{r}_{D/C}}_{\checkmark} \quad (3)$$

Plug in (1) in (2), (2) in (3)

$$\bar{v}_D = \bar{v}_B + \omega_{BC}\hat{k} \times \bar{r}_{C/B} + \omega_{CD}\hat{k} \times \bar{r}_{D/C}$$

$$\bar{0} = \text{from (1)} + \quad '' \quad \quad \quad '' \quad (4)$$

2 scalar unknowns. ω_{BC} & ω_{CD}.

1 vector equ. $\Rightarrow$ 2 scalar unknown.

$\hat{i}$ equation
$\hat{j}$ equation $\Bigg\}$ $\Rightarrow$ solve for ω_{BC} & ω_{CD}.

(b) EXACTLY same.

$$\bar{a}_B = \cancel{\bar{a}_A}^{0} + \alpha_{AB}\hat{k} \times \bar{r}_{B/A} - \omega_{AB}^2 \bar{r}_{B/A} \quad (5)$$

$$\bar{a}_C = \bar{a}_B + \alpha_{BC}\hat{k} \times \bar{r}_{C/B} - \omega_{BC}^2 \bar{r}_{C/B} \quad (6)$$

$$\cancel{\bar{a}_D}^{0} = \bar{a}_C + \alpha_{CD}\hat{k} \times \bar{r}_{D/C} - \omega_{CD}^2 \bar{r}_{D/C} \quad (7)$$

Plug (5) in (6), then (6) in (7).

everything known except α_{BC} & α_{CD}.
2 scalars.

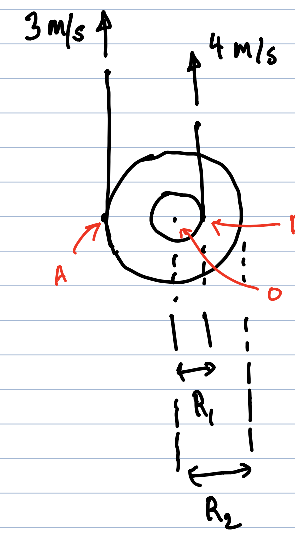
1 vector equ. $\Rightarrow$ 2 scalar equs

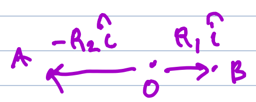
$\hat{i}$ equ
$\hat{j}$ equ. $\Big\}$ SOLVE

Rigid body kinematics (continued).

Another example.

pulley - - single rigid body





Given: a composite pulley with radii $R_1 = 1m$, $R_2 = 2m$. is a single rigid body. Two different ropes are attached to the 2 radii, and being pulled up with different velocities, as shown, WITH NO SLIP

Find :- angular velocity $\bar{\omega} = \omega\hat{k}$ of pulley

- velocity v_o of point O (center of pulley).

Consider B

$$\bar{v}_B = 4\hat{j} \ (m/s).$$

$$\bar{v}_B = \bar{v}_o + (\omega\hat{k} \times \bar{r}_{B/o})$$

$$4\hat{j} = \bar{v}_o + \omega\hat{k} \times (R_1\hat{i})$$

$$4\hat{j} = \bar{v}_o + R_1\omega\hat{j}$$

$$\boxed{\bar{v}_o = (4 - R_1\omega)\hat{j}} \quad (1)$$

Consider A

$$\bar{v}_A = 3\hat{j} \ (m/s).$$

$$\bar{v}_A = \bar{v}_o + \omega\hat{k} \times \bar{r}_{A/o}$$

$$3\hat{j} = \bar{v}_o + \omega\hat{k} \times (-R_2\hat{i})$$

$$= \bar{v}_o + (-R_2\omega)\hat{j}$$

$$\boxed{\bar{v}_o = (3 + R_2\omega)\hat{j}} \quad (2)$$

Need to solve for $\bar{v}_o$ and ω.

Equate $\bar{v}_o$ from (1) & (2).

$$(4 - R_1\omega)\cancel{\hat{j}} = (3 + R_2\omega)\cancel{\hat{j}}$$

$$4 - R_1\omega = 3 + R_2\omega$$

$$\omega(R_1 + R_2) = 4 - 3 = 1.$$

$$\omega = 1/(R_1 + R_2) = 1/(1+2) = 1/3 \text{ rad/s}$$

$$\boxed{\omega = 1/3 \text{ rad/s}}$$

Using equation (2).

$$\bar{V}_o = (3 + R_2\omega)\hat{j} = \left(3 + 2 \cdot \frac{1}{3}\right)\hat{j}$$

$$= \left(3 + \frac{2}{3}\right)\hat{j} = 3.67\,\hat{j} \text{ (m/s)}$$

$$\boxed{\bar{V}_o = 3.67\,\hat{j} \text{ (m/s)}}$$

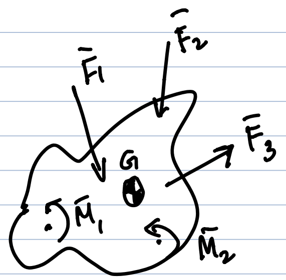
END.

2D RIGID BODY KINETICS

different from Kinematics.

relation between forces & motion.

Planar



Two main equations

① Newton's second law

$$\underbrace{\Sigma \bar{F}_i}_{\text{sum of all forces on rigid body}} = m \, \bar{a}_G$$

m: mass of rigid body

$\bar{a}_G$: acceleration of the G, the center of mass of the rigid body.

② Euler's equation.

$$\underbrace{\Sigma \bar{M}_{/G}}_{\text{sum of all moments about center of mass G}} = \underbrace{I_G}_{} \bar{\alpha} = I_G \alpha \hat{k}$$

I_G = "mass moment of inertia about an axis through G".

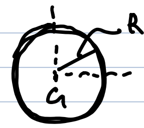
$\bar{\alpha} = \alpha \hat{k}$ - angular acceleration

Moment of inertia : mass moment of inertia

$$I_G = \int_{\text{whole object}} r^2 \, dm.$$

$\boxed{\text{different}}$ from the Area moment of inertia from "mechanics of materials" in beam bending

Eg 1



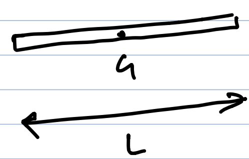
Ring of mass m, radius R
($R_{in} \approx R_{outer} = R$)

$$I_G = mR^2$$

Eg 2 Disk of mass m, radius R

$$I_G = \frac{mR^2}{2}.$$

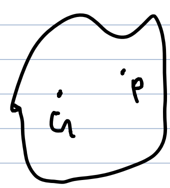
Eg 3



uniform bar of length L and mass m, but infinitesimal width

$$I_G = \frac{mL^2}{12}.$$

Parallel axis theorem : Allows computing the moment of inertia about a different point P.



$$\underbrace{I_P}_{\substack{\text{moment}\\\text{of inertia}\\\text{about } P}} = \underbrace{I_G}_{\substack{\text{moment}\\\text{of inertia}\\\text{about } G}} + m|GP|^2$$

Radius of gyration of a rigid body

By definition, $I_G = m.\begin{pmatrix}\text{radius of}\\\text{gyration}\end{pmatrix}^2.$

More useful version of Euler's equation

Original: $\sum \bar{M}_{/G} = I_G \alpha \hat{k}$.

taking moments about G.

Taking moments about a different point P.

③ $$\boxed{\sum \bar{M}_{/P} = I_G \alpha \hat{k} + \bar{r}_{G/P} \times m \bar{a}_G}$$

If point P is fixed on the rigid body & does not move, then

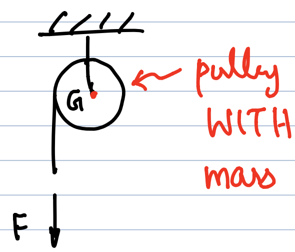
④ $$\boxed{\sum \bar{M}_{/P} = I_P \alpha \hat{k}}.$$

not true in general.

RIGID BODY KINETICS

Examples

Eg 1



Force $F = 100\,N$

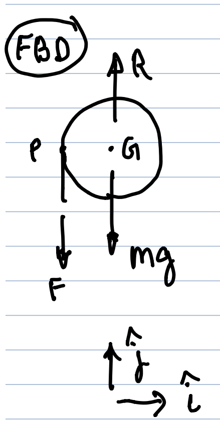
pulley mass $m = 1\,kg$.

pulley radius $R = 0.1\,m$.

The pulley is a uniform disk.

Find: the angular acceleration $\bar{\alpha} = \alpha\hat{k}$ of the pulley due to force F.

FBD



$$\Sigma \bar{M}_{/G} = I_G \alpha \hat{k}.$$

$$\bar{M}_{/G} \text{ of } F = FR\hat{k}. \leftarrow \text{total Moment about } G.$$

$$FR\cancel{\hat{k}} = I_G \alpha \cancel{\hat{k}}$$

$$\alpha = \frac{FR}{I_G} \leftarrow \text{moment of inertia of pulley}$$

$$I_G = \frac{mR^2}{2}$$

$$\alpha = \frac{FR}{(mR^2/2)} = \frac{2F}{mR} \Rightarrow \boxed{\alpha = \frac{2F}{mR}} \text{ ANS}$$

angular acceleration of pulley.

computing $\bar{M}_{/G}$

$$P \xleftarrow{-R\hat{i}} G \qquad \rightarrow \hat{i}$$

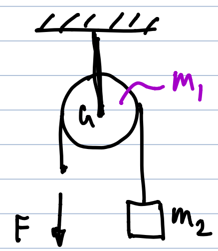
$$\bar{F} = -F\hat{j}. \qquad \bar{r}_{P/G} = -R\hat{i}$$

$$\bar{M}_{/G} = \bar{r}_{P/G} \times \bar{F} = (-R\hat{i}) \times (-F\hat{j}) = RF(\hat{i} \times \hat{j}) = FR\hat{k}.$$

cross product

RIGID BODY KINETICS

Examples



Given

pulley, mass m_1, radius R, uniform disk with moment of inertia $I_G = \frac{mR^2}{2}$.

rope is massless and inextensible.

Given: No slip between pulley & rope

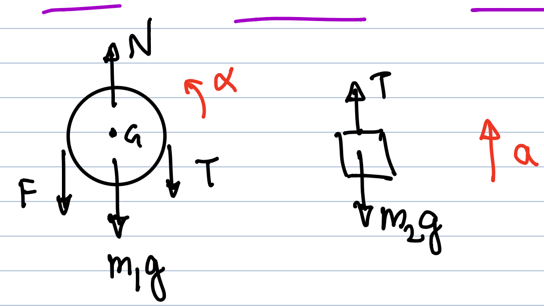
Find : a) - acceleration upward of m_2

b) - angular acceleration $\bar{\alpha} = \alpha \hat{k}$ of pulley.

c) tension in rope attached to m_2

d) reaction force between pulley and the ceiling

(1) Draw FBDs



(2) Use "$F = ma$" & "$\Sigma M = I_G \alpha$"

For pulley

$$\Sigma \bar{M}_{/G} = I_G \alpha \hat{k} = \frac{mR^2}{2} \alpha \hat{k}.$$

$$\Sigma \bar{M}_{/G} = FR\hat{k} - TR\hat{k}.$$

$$(FR - TR)\cancel{\hat{k}} = \frac{mR^2}{2} \alpha \cancel{\hat{k}}$$

$$(F - T)R = \frac{mR^2}{2}\alpha \Rightarrow \boxed{\alpha = \frac{2(F - T)}{mR}} \quad (1)$$

For pulley

For mass m_2

"$\sum F = m_1 a_G$"

$$\boxed{T - m_2 g = m_2 a} \quad (3)$$

(2) $N - m_1 g - F - T = m_1 . (0)$

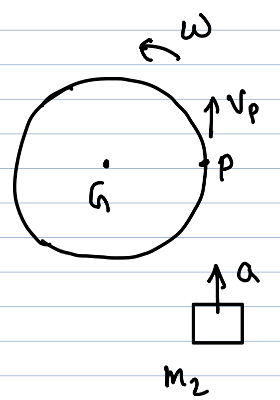
(2) $\boxed{N - m_1 g - F - T = 0}$

4 unknowns : α, a, T, N

3 equations so far.

If rope has NO SLIP relative to pulley.

then



$$v_G = 0$$

$$\bar{v}_P = \bar{v}_G + \bar{\omega} \times \bar{r}_{P/G}$$

$$= \bar{0} + \omega \hat{k} \times (R \hat{i})$$

$$v_P \hat{j} = R\omega \hat{j}$$

$$\boxed{v_P = R\omega}$$

$$a = a_P = R\alpha$$

No slip $\quad \boxed{a = R\alpha} \quad (4)$

(1) $\alpha = \dfrac{2(F-T)}{mR}$

(2) $T - m_2 g = m_2 a$

(3) $N - m_1 g - F - T = 0$

(4) $a = R\alpha.$

(1) $\Rightarrow \quad R\alpha = \dfrac{2(F-T)}{m_1}.$

(1)&(4) $\Rightarrow \quad a = \dfrac{2(F-T)}{m_1}. \quad (5)$

(2) $\Rightarrow \quad T = m_2 g + m_2 a.$

$$a = \frac{2(F - m_2 g - m_2 a)}{m_1}$$

$$m_1 a = 2(F - m_2 g - m_2 a)$$

$$m_1 a + 2m_2 a = 2(F - m_2 g)$$

ANS $$\boxed{a = \frac{2(F - m_2 g)}{m_1 + 2m_2}}$$

Using (4)

$$\alpha = \frac{a}{R}$$

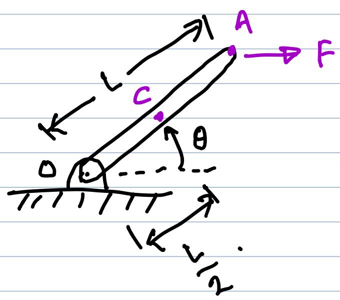
$$\boxed{\alpha = \frac{2(F - m_2 g)}{R(m_1 + 2m_2)}}$$

ANS

plug these results in (2) & (3) to compute T and then N.

DONE

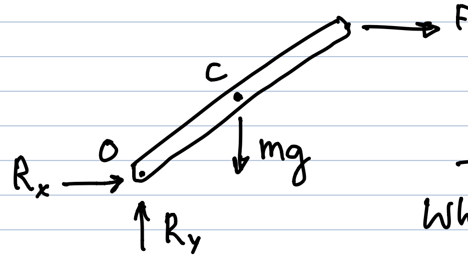
Example : Rigid body pendulum with a horizontal force



Find an expression for the angular acceleration $\alpha = \ddot{\theta}$ in terms of mass m of pendulum, length L, force F, $g \downarrow$ and angle θ. (i.e., determine the "equations of motion").

Assume OA : uniform bar, so that $I_c = mL^2/12$

FBD



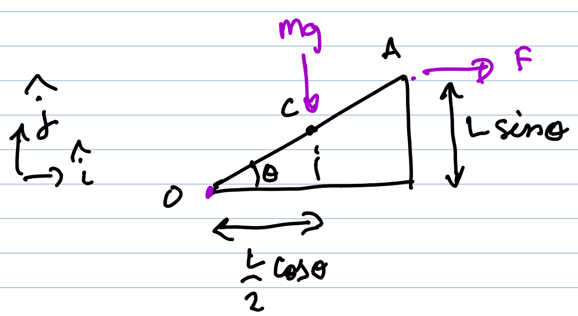
Recall $\Sigma \bar{M}_{/c} = I_c \alpha \hat{k}$. (1)

$$\Sigma \bar{M}_{/o} = I_c \alpha \hat{k} + \bar{r}_{c/o} \times m\bar{a}_c \quad (2)$$

Which one to use?

Using (1) means that we will get terms involving R_x & R_y in our equations.

To avoid solving for R_x and R_y, take moments about O. (so that R_x and R_y have no moments about O). Use eqn (2)

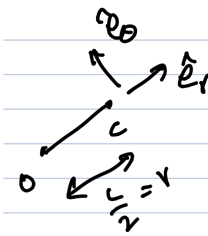


$$\Sigma \bar{M}_{/o} = -mg\frac{L}{2}\cos\theta\,\hat{k} - FL\sin\theta\,\hat{k} \quad (3)$$

We need to compute other terms in

$$\Sigma \bar{M}_{/o} = I_c \alpha \hat{k} + \bar{r}_{c/o} \times m\bar{a}_c$$

✓



$I_c = \frac{mL^2}{12}$, $\alpha = \ddot{\theta}$ (4)

$\bar{r}_{c/o} = \frac{L}{2}\cos\theta\,\hat{i} + \frac{L}{2}\sin\theta\,\hat{j} = \frac{L}{2}\hat{e}_r$

Recall polar coordinates

$\bar{a}_c = -r\dot{\theta}^2\,\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta$

$= -\frac{L}{2}\dot{\theta}^2\,\hat{e}_r + \frac{L}{2}\ddot{\theta}\,\hat{e}_\theta .$ (6)

$\bar{r}_{c/o} \times m\bar{a}_c = m\frac{L}{2}\hat{e}_r \times \left[-\frac{L}{2}\dot{\theta}^2\,\hat{e}_r + \frac{L}{2}\ddot{\theta}\,\hat{e}_\theta\right]$

$= \bar{0} + m\left(\frac{L}{2}\right)^2\ddot{\theta}\,(\hat{e}_r \times \hat{e}_\theta) = m\ddot{\theta}\left(\frac{L}{2}\right)^2 \cdot \hat{k} .$

$\Sigma\bar{M}_{/o} = I_c\alpha\,\hat{k} + \bar{r}_{c/o} \times m\bar{a}_c .$

$\left(-mg\frac{L}{2}\cos\theta - FL\sin\theta\right)\cancel{\hat{k}} = \frac{mL^2}{12}\ddot{\theta}\,\cancel{\hat{k}} + m\ddot{\theta}\left(\frac{L}{2}\right)^2\cancel{\hat{k}}$

$= \ddot{\theta}\left(\frac{mL^2}{12} + \frac{mL^2}{4}\right) = \ddot{\theta} \cdot \frac{mL^2}{3} .$

ANS

angular acceleration:

$$\boxed{\ddot{\theta} = \frac{-mg\frac{L}{2}\cos\theta - FL\sin\theta}{mL^2/3}} = \alpha$$

Method 2: Because O is a point fixed on the rigid body and in space, we can use

$\Sigma\bar{M}_{/o} = I_o\alpha\,\hat{k}$ (7)

as opposed to equ (2)

I_o = computed using the parallel axis theorem $= I_c + m\,|oc|^2$

$$= \frac{mL^2}{12} + m \cdot \left(\frac{L}{2}\right)^2$$

$$= \frac{mL^2}{3}.$$

Plug in equ (7).

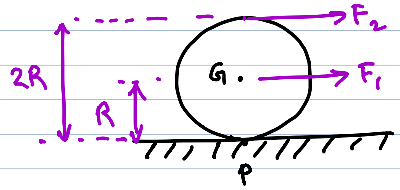
$$-mg\frac{L}{2}\cos\theta - FL\sin\theta = I_o\alpha$$

$$= \frac{mL^2}{3}\alpha$$

ANS

$$\boxed{\alpha = \frac{-mg\frac{L}{2}\cos\theta - FL\sin\theta}{\frac{mL^2}{3}}.}$$

Example.



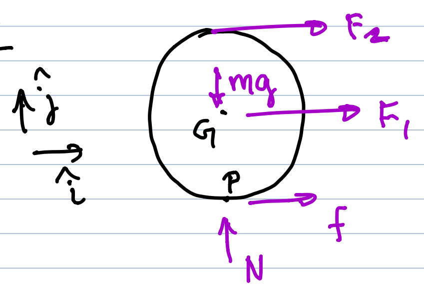
A uniform disk of mass m, radius R being pulled by forces F_1 & F_2, as shown.

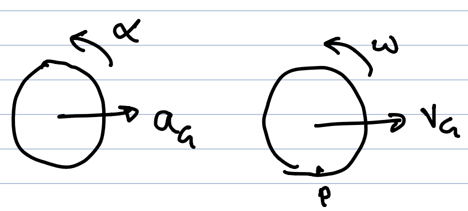
The disk rolls without slip on the floor.

Determine: a) the acceleration of point G.

b) the friction force at point P.

FBD





Rolling without slip

Recall from earlier that

$$\bar{v}_P = \bar{0} = v_G\hat{i} + \omega\hat{k}\times(\underbrace{-R\hat{j}}_{\bar{r}_{P/G}})$$

$$\bar{0} = (v_G + R\omega)\hat{i}$$

$$\Rightarrow \boxed{v_G = -R\omega} \text{ (or) } \boxed{a_G = -R\alpha} \quad (1)$$

"$\Sigma\bar{F} = m\bar{a}$" along $\hat{j}$

$$N - mg = m(0)$$

$$\boxed{N = mg} \quad (2)$$

"$\Sigma \bar{F} = m\bar{a}$" along $\hat{i}$

$$\boxed{F_1 + F_2 + f = m a_G} \quad (3)$$

$\Sigma \bar{M}_{/G} = I_G \alpha \hat{k}$

$$f R \hat{k} - F_2 R \hat{k} + F_1 (0) \hat{k} = I_G \alpha \hat{k}$$

$$(f R - F_2 R) \hat{k} = \frac{mR^2}{2} \alpha \hat{k} .$$

$$\boxed{f - F_2 = \frac{mR}{2} \alpha .} \quad (4)$$

Unknowns are: a_G, α, f, N.

Note: For rolling without slip,

friction force $\neq \mu_k N$

But is determined by solving equations (1) - (4).

Analogous to "static friction"

The maximum value of f such that the disk doesn't slip is $\mu_s N$.

Use (1) in (4) : $f - F_2 = \frac{m}{2}(R\alpha) = \frac{m}{2}(-a_G)$

$$f - F_2 = -\frac{m a_G}{2} \quad (5)$$

$$F_1 + F_2 + f = m a_G \quad (3)$$

$$\underline{(-) \qquad\qquad (-)}$$

(5) - (3)

$$\cancel{f} - F_2 - F_1 - F_2 - \cancel{f} = -\frac{m a_G}{2} - m a_G$$

$$+2F_2 + F_1 = +\frac{3 m a_G}{2}.$$

$$(6) \quad \boxed{a_G = \frac{2}{3m}(2F_2 + F_1)} \quad \text{ANS (a)}$$

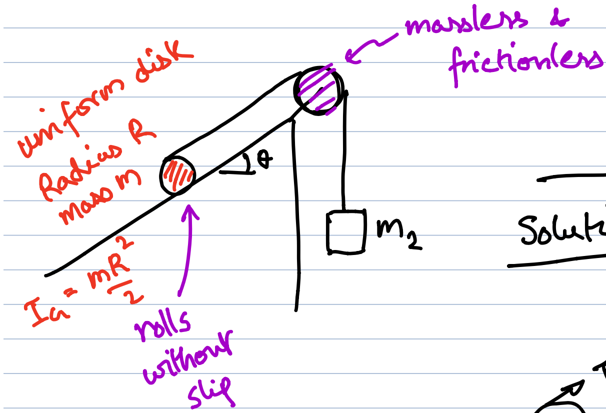
Use (6) in (5) to get

$$f = F_2 - \frac{m a_G}{2} = F_2 - \frac{\cancel{m}}{\cancel{2}} \cdot \frac{\cancel{2}}{3\cancel{m}}(2F_2 + F_1)$$

$$\boxed{f = F_2 - \frac{(2F_2 + F_1)}{3}.} \quad \text{ANS (b)}$$

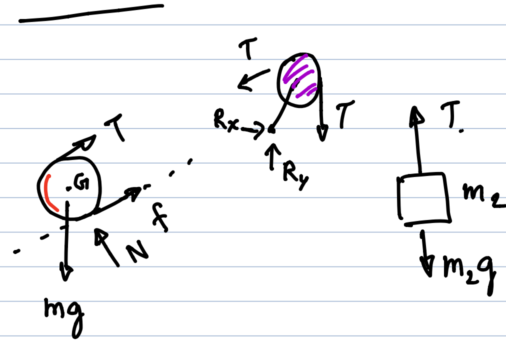
$$\alpha = -\frac{a_G}{R} = \boxed{-\frac{2(2F_2 + F_1)}{3mR} = \alpha}$$

Example : More complex pulley problem

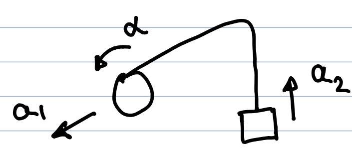


Compute the acceleration of m_2 in terms of m, R, m_2, g.

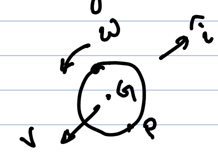
Solution



Sign conventions



Rolling without slip



$$\bar{v}_p = \bar{v}_G + \omega\hat{k} \times \underbrace{(-R\hat{j})}_{\bar{r}_{P/G}}$$

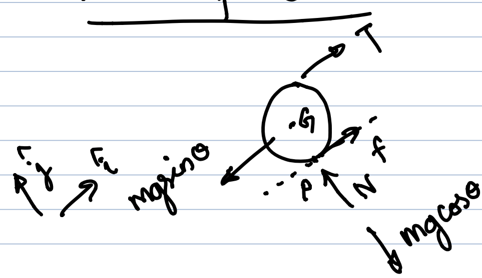
$$\bar{0} = -v\hat{i} - R\omega(-\hat{i})$$

$$-v + R\omega = 0$$

$$\boxed{v = R\omega} \quad (1) \Rightarrow \boxed{a_1 = R\alpha}$$

"For m_2" : $\boxed{m_2 a_2 = T - m_2 g}$ (2)

F = ma for disk



along $\hat{i}$

$$\boxed{ma_1 = mg\sin\theta - T - f} \quad (3)$$

along $\hat{j}$

$$\boxed{N - mg\cos\theta = m(0)} \quad (4)$$

$$\underline{\Sigma \bar{M}_{/a}} = I_a \alpha \hat{k}$$

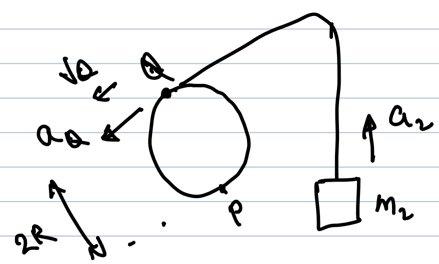
$$fR\cancel{\hat{k}} - TR\cancel{\hat{k}} = \frac{mR^2}{2}\alpha \cancel{\hat{k}} \Rightarrow (f-T)R = \frac{mR^2}{2}\alpha$$

$$\boxed{f - T = \frac{mR\alpha}{2}} \quad (5)$$

6 <u>unknowns</u> : $a_1, \alpha, T, a_2, f, N$

need one more equation.

relation between a_2 and α.



$$\left.\begin{aligned} v_Q &= 2R\omega \\ a_Q &= 2R\alpha \end{aligned}\right] \Rightarrow a_Q = a_2$$

$$\Rightarrow \boxed{2R\alpha = a_2} \quad (6)$$

$$\boxed{\alpha = \frac{a_2}{2R}} \quad (6)$$

Use (6) in (5) $\Rightarrow f - T = \frac{mR\alpha}{2} = \frac{mR}{2}\frac{a_2}{2R} = \frac{ma_2}{4}$

$$\boxed{f - T = \frac{ma_2}{4}} \quad (7) \qquad \boxed{\frac{ma_2}{4} = f - T} \quad (7)$$

Use (6) in (1) $\Rightarrow a_1 = R\alpha = R \cdot \frac{a_2}{2R} = \frac{a_2}{2}$

$$\boxed{a_1 = \frac{a_2}{2}} \quad (8)$$

Use (8) in (3) :

$$ma_1 = mg\sin\theta - T - f \quad (3)$$

$$\boxed{\frac{m a_2}{2} = mg\sin\theta - T - f} \quad (9)$$

$$\frac{M a_2}{4} = f - T \quad (7)$$

$$(7) + (9) \Rightarrow \frac{m a_2}{2} + \frac{m a_2}{4} = mg\sin\theta - T - \cancel{f} + \cancel{f} - T$$

$$\boxed{\frac{3 m a_2}{4} = mg\sin\theta - 2T} \quad (10)$$

$$(2) \Rightarrow m_2 a_2 = T - m_2 g$$

$$2 m_2 a_2 = 2T - 2m_2 g \quad (11)$$

Add $(10) + (11)$

$$\frac{3}{4} m a_2 + 2m_2 a_2 = mg\sin\theta - \cancel{2T} + \cancel{2T} - 2m_2 g$$

$$\left(\frac{3m}{4} + 2m_2\right) a_2 = mg\sin\theta - 2m_2 g$$

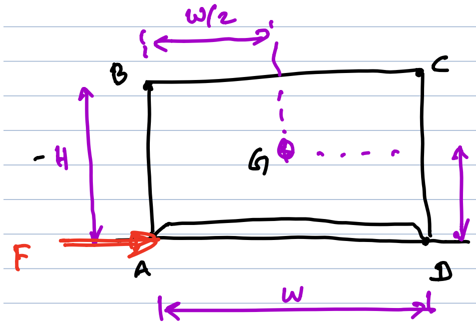
$$\boxed{a_2 = \frac{mg\sin\theta - 2m_2 g}{\frac{3m}{4} + 2m_2}} \quad \text{ANS}$$

To see if the disk will slip, solve for f

& check whether $|f| > \mu_s N$ if yes, slip

if no, no slip.

Example : Accelerating car or block



No friction at A and D

Given Block ABCD. mass $m = 1000$ kg.

$g = 9.81\ m/s^2$.

$I_G = \dfrac{m(H^2 + W^2)}{12} = $ —

$H = 1m,\ W = 2m.$

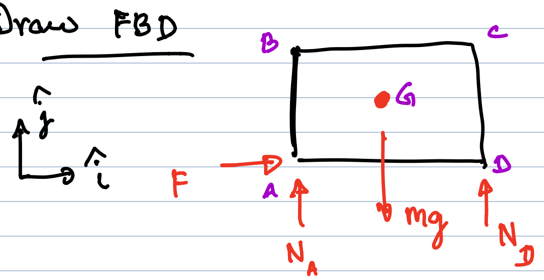
a) Force $F = 500$ N

b) Force $F = 100{,}000$ N.

Compute: (i) The normal reactions at A and D

(ii) The acceleration of point G.

Draw FBD



"acceler ation" $\uparrow 0$ no movement vertically,

$\rightarrow a_G$ (say)

$\Sigma \bar{F} = m\bar{a}_G$

$(\hat{i})$: $F = m a_G$.

$\boxed{a_G = F/m}$ ANS(ii) (1)

$(\hat{j})$: $N_A + N_D - mg = m(0)$

$\boxed{N_A + N_D = mg}$ (2)

$\Sigma \bar{M}_{/G} = I_G \alpha \hat{k}$.

$\alpha = 0$ if moving horizontally without rotation.

$\left[F \cdot \frac{H}{2}\hat{k} + N_D \cdot \frac{W}{2}\hat{k} - N_A \cdot \frac{W}{2}\hat{k}\right] = I_G \alpha \hat{k} = \bar{0}$

$\Rightarrow F \cdot \frac{H}{\cancel{2}} + N_D \frac{W}{\cancel{2}} - N_A \frac{W}{\cancel{2}} = 0 \Rightarrow \boxed{N_D - N_A = \frac{FH}{W}}$ (3)

$(2) + (3)$: $\cancel{N_A} + N_D + N_D - \cancel{N_A} = mg - \frac{FH}{w}$

$$2N_D = mg - \frac{FH}{w}$$

Ans (ii) $\boxed{N_D = \frac{1}{2}\left[mg - \frac{FH}{w}\right]}$. (4)

Use (2). $N_A + N_D = mg$.

$$N_A + \frac{1}{2}mg - \frac{1}{2}\frac{FH}{w} = mg.$$

$$N_A = \frac{FH}{2w} + \frac{mg}{2} = \frac{1}{2}\left[mg + \frac{FH}{w}\right]$$

Ans ii $\boxed{N_A = \frac{1}{2}\left[mg + \frac{FH}{w}\right]}$ (5)

(a) $m = 1000$ kg. $g = 9.81\ m/s^2$. $F = 500\ N$, $H = 1\ m$, $W = 2\ m$.

$$N_A = \frac{1}{2}\left[9810 + \frac{500 \cdot 1}{2}\right] = \frac{1}{2}\left[9810 + 250\right]\ N$$

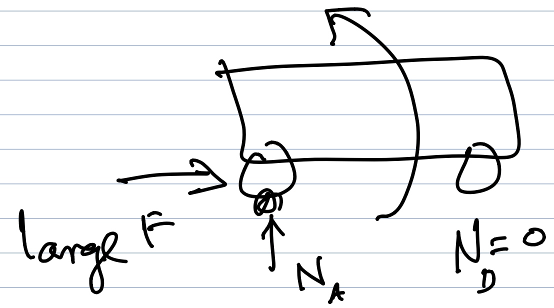
$$N_D = \frac{1}{2}\left[9810 - \frac{500}{2}\right] = \frac{1}{2}\left[9810 - 250\right].\ N.$$

(b) $F = 100\,000\ N$.

$$N_A = \frac{1}{2}\left[9810 + \frac{10^4}{2}\right] = \frac{1}{2}\left[9810 + 50\,000\right]\ N$$

$N_D = \frac{1}{2}\left[9810 - 50000\right]$ = negative normal reaction.

"WHEELIE" youtube. Do a search.

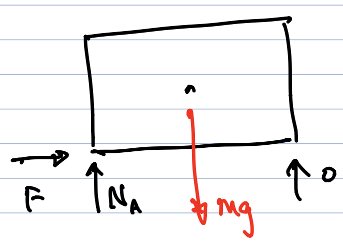


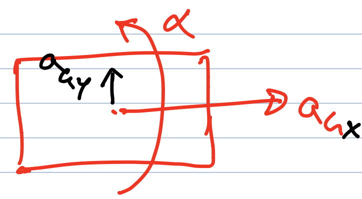
part-b re-do

Okay N_D cannot be negative.

N_D can at best be 0.

$N_D = 0$. The object pivots about A.





$\hat{i}$: $F = m a_{Gx}$ (1)

$\hat{j}$: $N_A - mg = m a_{Gy}$. (2)

$\sum \bar{M}_{/G} = I_G \alpha \hat{k}$.

$$\frac{FH}{2} + \cancel{N_D}^{0} \frac{W}{2} - N_A \frac{W}{2} = I_G \alpha. \quad (3)$$

unknown : a_{Gx}, a_{Gy}, α, N_A.

extra equation : $\bar{a}_A$ is only along $\hat{i}$ and has zero component along $\hat{j}$.

$$\bar{a}_A = \bar{a}_G - \cancel{\omega^2 \bar{r}_{A/G}}^{\,\to 0} + \alpha \hat{k} \times \bar{r}_{A/G}$$

omega

$$\bar{a}_A = a_{Gx}\hat{i} + a_{Gy}\hat{j} + \alpha\hat{k} \times \left(-\frac{W}{2}\hat{i} - \frac{H}{2}\hat{j}\right)$$

$$= a_{Gx}\hat{i} + a_{Gy}\hat{j} + \left(-\frac{\alpha W}{2}\right)\hat{j} - \frac{H\alpha}{2}\left(-\hat{i}\right)$$

$$\bar{a}_A = \hat{i}\left(a_{Gx} + \frac{H\alpha}{2}\right) + \hat{j}\left(a_{Gy} - \frac{\alpha W}{2}\right).$$

$\bar{a}_A$ only along $\hat{i}$ and not along $\hat{j}$

$\Rightarrow$ set $\hat{j}$ component to zero

$$a_{Gy} - \frac{\alpha W}{2} = 0$$

$$\boxed{a_{Gy} = \frac{\alpha W}{2}} \quad (4)$$

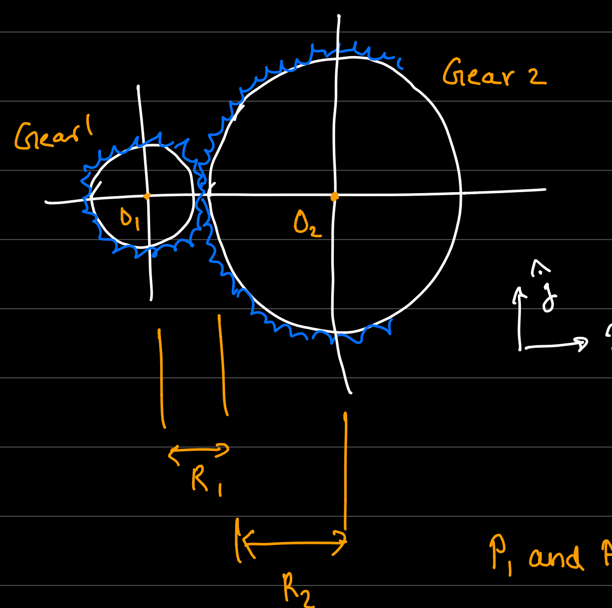
You can now solve for the 4 unknowns a_{Gx}, a_{Gy}, α, and N_A using these new (1) – (4).

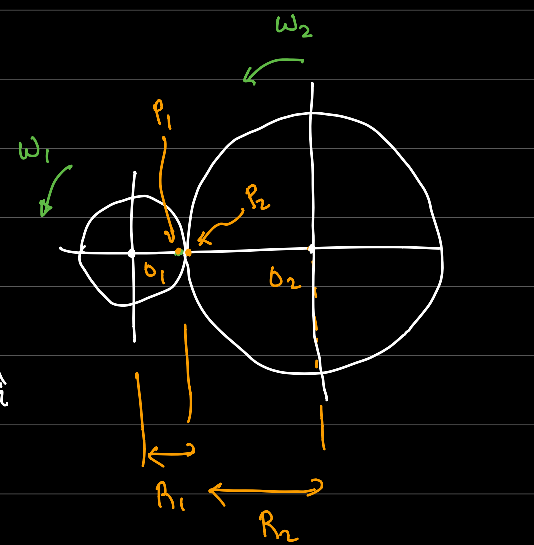
THE END

We will leave this as an exercise.

A couple of transmission systems

Gears





P_1 and P_2 are points on gears 1 and 2 that contact each other.

$\bar{V}_{O1} = 0$, $\bar{V}_{O2} = 0$

O_1 and O_2 are fixed in space.

What is the relation between ω_1 and ω_2?

- the gears do not slip against each other
- No slip $\Rightarrow$ $\boxed{\bar{V}_{P1} = \bar{V}_{P2}}$ (1)

$\boxed{\bar{V}_{P1} = R_1\omega_1\hat{j}}$ (2)

$$\bar{V}_{P1} = \bar{V}_{O1} + \omega_1\hat{k} \times \bar{r}_{P1/O1}$$
$$= \bar{0} + \omega_1\hat{k} \times (R_1\hat{i})$$
$$= R_1\omega_1\hat{j}$$

reason for (2)

$\bar{V}_{P2} = R_2\omega_2(-\hat{j}) = -R_2\omega_2\hat{j}$

$\boxed{V_{P2} = -R_2\omega_2\hat{j}}$ (3)

reason for (3)

$$\bar{V}_{P2} = \bar{V}_{O2} + \omega_2\hat{k} \times \bar{r}_{P2/O2}$$

$$= \bar{0} + \omega_2 \hat{k} \times (-R_2 \hat{i})$$

$$= -\omega_2 R_2 (+\hat{j})$$

$$\boxed{V_{P_2} = -R_2 \omega_2 \hat{j}}$$

$$R_1 \omega_1 \cancel{\hat{j}} = -R_2 \omega_2 \cancel{\hat{j}}$$

speed ratio $\boxed{\frac{\omega_1}{\omega_2} = \frac{-R_2}{R_1}}$ ← ratio of radii = "gear ratio".

$$\Rightarrow \boxed{\frac{\alpha_1}{\alpha_2} = \frac{-R_2}{R_1}}$$

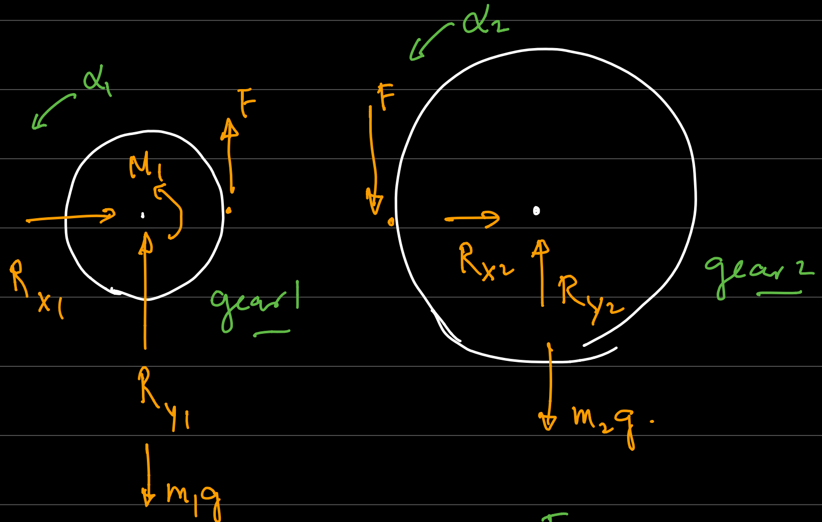
Say a motor at O_1 was applying a moment (torque) $M_1 \hat{k}$ at O_1, where M_1 value is known.

The gears are rigid bodies with moments of inertia I_1 and I_2 respectively about O_1 and O_2 respectively.

Determine the angular accelerations α_1 and α_2 of the 2 gears.

Gear 2 has no friction or any torque at O_2.

FBDs of the gears (ignore gravity)



For gear 1

$$\sum \bar{M}_{O_1} = I_1 \alpha_1 \hat{k}$$

$$\boxed{M_1 + F R_1 = I_1 \alpha_1} \quad (1)$$

For gear 2

$$\boxed{F R_2 = I_2 \alpha_2} \quad (2)$$

(1) and (2) have 3 unknowns α_1, α_2, F

Need a 3^{rd} eqn.

$$\boxed{\frac{\alpha_1}{\alpha_2} = -\frac{R_2}{R_1}} \quad (3)$$

From (2), $F = I_2 \alpha_2 / R_2$, use in (1)

$$(1) \Rightarrow M_1 + \left(\frac{I_2 \alpha_2}{R_2}\right) R_1 = I_1 \alpha_1 \quad (4)$$

$$(3) \Rightarrow \alpha_1 = -\frac{\alpha_2 R_2}{R_1}$$

$$M_1 + \left(\frac{I_2 \alpha_2}{R_2}\right) R_1 = I_1 \left(\frac{-\alpha_2 R_2}{R_1}\right)$$

$$M_1 + \left(\frac{I_2 R_1}{R_2}\right)\alpha_2 + \alpha_2 \left(\frac{I_1 R_2}{R_1}\right) = 0$$

$$M_1 = -\alpha_2 \left[\frac{I_2 R_1}{R_2} + \frac{I_1 R_2}{R_1}\right] = -\alpha_2 \left[\frac{I_2 R_1^2 + I_1 R_2^2}{R_1 R_2}\right].$$

$$\boxed{\alpha_2 = \frac{-M_1 \cdot R_1 R_2}{I_2 R_1^2 + I_1 R_2^2}}$$ ANS

$$\alpha_1 = -\frac{\alpha_2 R_2}{R_1} = -\left[\frac{-M_1 \cdot \cancel{R_1} R_2}{I_2 R_1^2 + I_1 R_2^2}\right] \cdot \frac{R_2}{\cancel{R_1}}$$

$$= \frac{M_1 \cdot R_2^2}{I_2 R_1^2 + I_1 R_2^2} = \frac{M_1}{I_2 \frac{R_1^2}{R_2^2} + I_1}$$

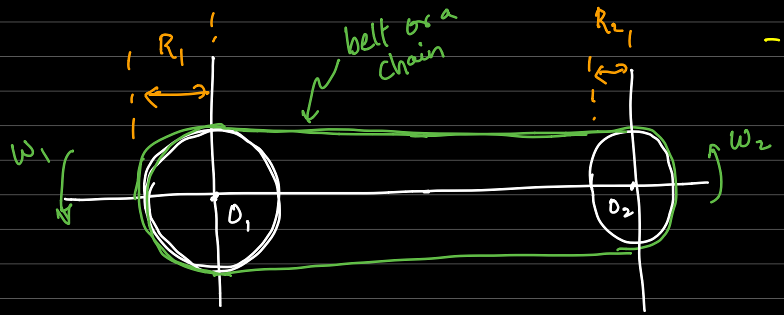
$$\boxed{\alpha_1 = \frac{M_1}{I_1 + \frac{I_2 R_1^2}{R_2^2}}}$$ ANS

extra term.

reflected inertia of gear 2.

another kind of transmission

Belt drive / chain drive

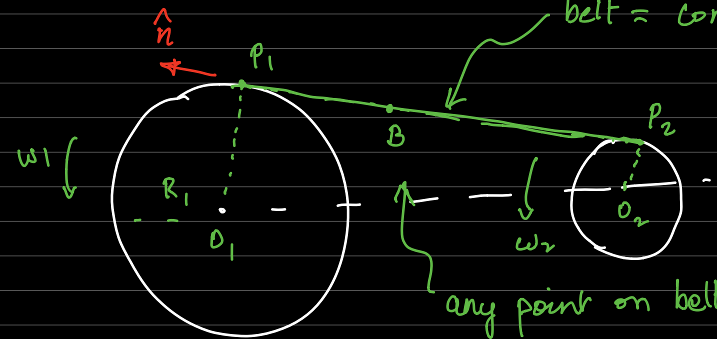


- belt / chain is inextensible and massless and is always taut (is in tension) and does NOT slip against the pulleys

pulley 1
uniform disk.
m_1, R_1
$I_{O_1} = \frac{m_1 R_1^2}{2}$

pulley 2
uniform disk
mass m_2 radius R_2
$I_{O_2} = \frac{m_2 R^2}{2}$.

Part 1: Say the pulley 1 has angular velocity ω_1. what is the angular velocity of pully 2 in terms of ω_1, R_1 and R_2.



belt = common tangent to the 2 circles.

any point on belt between P_1 and P_2

P_1 and P_2 are where the belt is tangent to the circles

No slip $= \bar{V}_{belt} = \bar{V}_B = \bar{V}_{P_1} = \bar{V}_{P_2}$

ricos

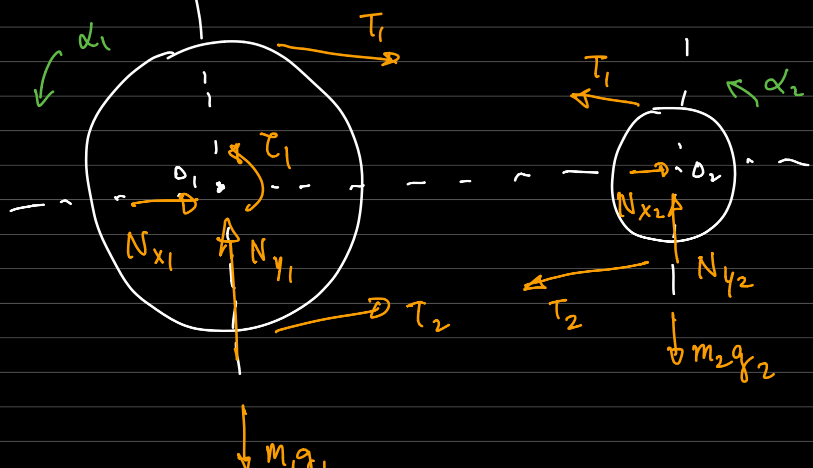
$\bar{V}_{P_1} = R_1 \omega_1 \hat{n}$, $\bar{V}_{P_2} = R_2 \omega_2 \hat{n}$ kinema.

$R_1 \omega_1 = R_2 \omega_2$

$$\frac{\omega_2}{\omega_1} = \frac{R_1}{R_2} \Rightarrow \boxed{\omega_2 = \omega_1 \frac{R_1}{R_2}}$$ ANS

Part 2. Pulley 1, is acted upon by torque $\tau_1 \hat{k}$ at center O_1. No torque at axle O_2 on pulley 2.

Determine the angular accelerations of pulleys 1 and 2.



$\Sigma \bar{M}_{/O_1} = I_{O_1} \alpha_1 \hat{k}$.

$\cancel{\hat{k}}(\tau_1 - T_1 R_1 + T_2 R_1) = I_{O_1} \alpha_1 \cancel{\hat{k}}$

(1) $\tau_1 - T_1 R_1 + T_2 R_1 = I_{O_1} \alpha_1$

$\Sigma \bar{M}_{/O_2} = I_{O_2} \alpha_2 \hat{k}$

$(T_1 R_2 - T_2 R_2)\cancel{\hat{k}} = I_{O_2} \alpha_2 \cancel{\hat{k}}$

(2) $(T_1 - T_2) R_2 = I_{O_2} \alpha_2$

(1) $\tau_1 - (T_1 - T_2) R_1 = I_{01} \alpha_1.$

Unknowns: $\alpha_1, \alpha_2, T_1, T_2$. 4 unknowns

$$\boxed{\alpha_1 R_1 = \alpha_2 R_2} \quad (3)$$

unknowns : α_1, α_2, Q

where $Q = T_1 - T_2$

(1) $\tau_1 - QR_1 = I_{01} \alpha_1$

(2) $QR_2 = I_{02} \alpha_2$

(3) $\alpha_1 R_1 = \alpha_2 R_2$

solve for $\underbrace{\alpha_1, \alpha_2}, Q$.

$$\frac{(1)}{R_1} + \frac{(2)}{R_2}$$

$$\frac{\tau_1}{R_1} - Q = \frac{I_{01} \alpha_1}{R_1}$$

$$+ \quad Q = \frac{I_{02} \alpha_2}{R_2}$$

$$\frac{\tau_1}{R_1} = \frac{I_{01} \alpha_1}{R_1} + \frac{I_{02} \alpha_2}{R_2}$$

$$= \frac{I_{01} \alpha_1}{R_1} + \frac{I_{02}}{R_2} \cdot \left(\frac{\alpha_1 R_1}{R_2} \right)$$

$$= \alpha_1 \left[\frac{I_{o1}}{R_1} + \frac{I_{o2} R_1}{R_2^2} \right] = \alpha_1 \left[\frac{m_1 R_1^2}{2R_1} + \frac{m_2 R_2^2}{2R_2} \cdot \frac{R_1}{R_2} \right]$$

$$\frac{\tau_1}{R_1} = \frac{\alpha_1}{2} \left[m_1 R_1 + m_2 R_1 \right]$$

$$\Rightarrow \boxed{\alpha_1 = \frac{2\tau_1}{m_1 R_1^2 + m_2 R_1^2}}$$ ANS

$$\alpha_2 = \frac{\alpha_1 R_1}{R_2} = \boxed{\frac{2\tau_1}{(m_1 + m_2) R_1^2} \cdot \frac{R_1}{R_2}}$$ ANS

Work-energy methods: Rigid bodies

Recall:

$$KE_i + PE_i + \text{Work done by other forces} = KE_f + PE_f .$$

$$(KE_f + PE_f) - (KE_i + PE_i) = \text{work.}$$

$$(KE_f - KE_i) + (PE_f - PE_i) = \text{work by other forces (not gravity / springs)}$$

STILL TRUE if we have rigid bodies

- Can use these for the whole system or just a single FBD.

For a 2D rigid body

$$\bar{v}_G = v_x \hat{i} + v_y \hat{j}$$

translational KE

rotational KE.

$$KE = \overbrace{\frac{1}{2} m \underbrace{\left(v_x^2 + v_y^2\right)}_{|\bar{v}_G|^2}}^{\text{translational KE}} + \overbrace{\frac{1}{2} I_G \omega^2}^{\text{rotational KE}}$$

PE due to gravity $\Big] = m g \underbrace{y_G}$

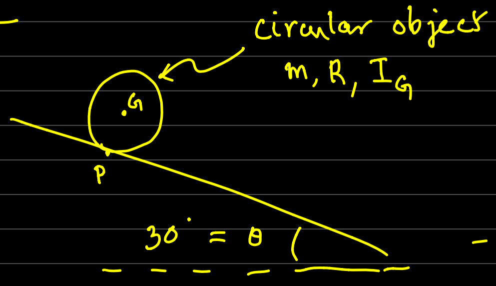
vertical position of the center of mass.

Work of a force $\bar{F}$ acting at P $= \int \bar{F} \cdot d\bar{r}_P = \int \underbrace{\left(\bar{F} \cdot \bar{v}_P\right)}_{\text{power}} dt$

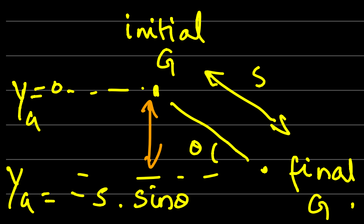
work of a moment M $= \int M \, d\theta = \int M \omega \, dt$

$M \, d\theta = M \frac{d\theta}{dt} dt$

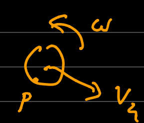
Example



- Object rolls without slip.
- Starts at rest.
- Determine the velocity of G when G has traveled a distance s along the slope



(1) $KE_i + PE_i +$ Work done by other forces (not gravity or springs) $= KE_f + PE_f$.



$KE_i = 0$ (rest)

$PE_i = 0$ (datum)

$KE_f = ? = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

Rolling without slip.

$\boxed{v_G = -R\omega}$

$PE_f = mg\, y_{Gf}$

$PE_f = mg(-s \cdot \sin\theta)$.

$KE_f = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \cdot \left(\frac{-v_G}{R}\right)^2$

$KE_f = \frac{1}{2} v_G^2 \left(m + \frac{I_G}{R^2}\right)$.

What about work by normal reaction or friction?

$W = \int \left(\bar{F} \cdot \bar{v}_p\right) dt$ p is point of application of force.

Both friction + normal reaction act at P

& $\bar{v}_P = \bar{0}$ (no slip)

$\Rightarrow W = 0$

Using everything in ①.

$$0 + 0 + 0 = \frac{1}{2}\left(m + \frac{I_G}{R^2}\right)v_G^2 - mgs.\sin\theta.$$

$$v_G^2 = \frac{2mgs.\sin\theta}{\left(m + \frac{I_G}{R^2}\right)}.$$

$$\boxed{v_G = \sqrt{\frac{2mgs.\sin\theta}{m + I_G/R^2}}}$$ ANS

Case 1 $I_G = 0$. $\Rightarrow$ particle. (no friction)

$$v_G = \sqrt{\frac{2\cancel{m}gs.\sin\theta}{\cancel{m}}} = \sqrt{2gs.\sin\theta}$$

$$\boxed{v_G = \sqrt{2gs.\sin\theta}}$$

Case 2 uniform disk $I_G = \frac{mR^2}{2}$.

$$v_G = \sqrt{\frac{2mgs.\sin\theta}{m + I_G/R^2}} = \sqrt{\frac{2\cancel{m}gs.\sin\theta}{\cancel{m} + \frac{\cancel{m}\cancel{R^2}}{2\cancel{R^2}}}}$$

$$= \sqrt{\frac{2\ gs \cdot \sin\theta}{3/2}} = \sqrt{\frac{4}{3} gs \cdot \sin\theta}$$

$$\boxed{V_G = \sqrt{\frac{4}{3} gs \cdot \sin\theta}}$$

Case 3 ring. $I_G = mR^2$

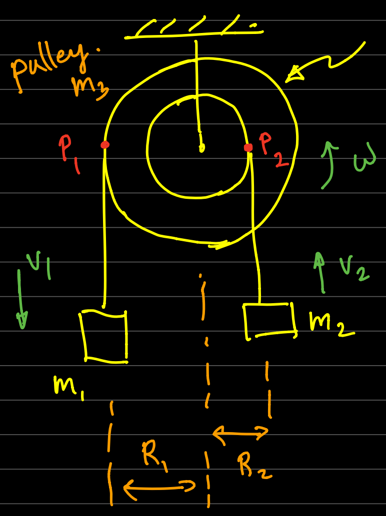
$$V_G = \sqrt{\frac{2mgs \cdot \sin\theta}{m + I_G/R^2}} = \sqrt{\frac{2\cancel{m}gs \cdot \sin\theta}{\cancel{m} + \frac{\cancel{mR^2}}{\cancel{R^2}}}}$$

$$V_G = \sqrt{\frac{\cancel{2}gs \cdot \sin\theta}{\cancel{2}}} = \sqrt{gs \cdot \sin\theta}$$

$$\boxed{V_G = \sqrt{1 \cdot gs \cdot \sin\theta}}$$

Example : Pulley problem with energy methods

no slip between pulley & rope



composite pulley with a single rigid body.

Pulley is a uniform disk of radius R_1

$I_G = \dfrac{m_3 R_1^2}{2}$.

Q: Say the system starts at rest.

Determine the angular velocity ω of the pulley when mass m_1 has descended by height H.

→ use energy methods. ←

① $KE_i + PE_i +$ work by other forces (not gravity or springs) $= KE_f + PE_f$.

② $KE_i = 0$ (rest)

③ $PE_i = 0$ (datum).

④ $KE_f = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 + \frac{1}{2} I_G \omega^2$.

In ④, can we replace v_{1f} and v_{2f} in terms of ω?

Yes. → $\underbrace{v_1 = v_{P_1}}_{\text{no slip}} \underbrace{= R_1\omega}_{P_1 \text{ on rigid body pulley}}$ ⑤

Similarly, $v_2 = v_{P2} = R_2\omega$. ⑥

$$KE_f = \frac{1}{2} m_1 (R_1 \omega_f)^2 + \frac{1}{2} m_2 (R_2 \omega_f)^2 + \frac{1}{2} \frac{m_3 R_1^2}{2} \cdot \omega_f^2$$

$$KE_f = \frac{1}{2} \omega_f^2 \left[m_1 R_1^2 + m_2 R_2^2 + \frac{m_3 R_1^2}{2} \right]. \quad (5)$$

Work by other forces $= 0$.

$$PE_f = m_1 g\, y_{1f} + m_2 g\, y_{2f}$$

$$= m_1 g (-H) + m_2 g\, y_{2f}$$

what is y_{2f}? (measured from its own initial position)

m_1 goes down by H. (given).

$$\left.\begin{aligned} v_1 &= R_1 \omega \\ v_2 &= R_2 \omega \end{aligned}\right] \Rightarrow \frac{v_2}{v_1} = \frac{R_2}{R_1} \Rightarrow v_2 = \frac{R_2}{R_1} v_1$$

m_2 goes UP by $H \cdot \frac{R_2}{R_1}$.

$$y_{2f} = H \cdot \frac{R_2}{R_1}.$$

$$PE_f = m_1 g (-H) + m_2 g \left(\frac{H R_2}{R_1} \right). \quad (6)$$

Plug in (5), (6), (2), (3) into (1)

$$0 + 0 + 0 = m_1 g(-H) + m_2 g\left(\frac{HR_2}{R_1}\right) + \frac{1}{2}\omega^2\left[m_1R_1^2 + m_2R^2 + \frac{m_3R_1^2}{2}\right]$$

$$\frac{1}{2}\omega^2 = \frac{m_1 g H - m_2 g HR_2/R_1}{\left[m_1R_1^2 + m_2R_2^2 + \frac{m_3R_1^2}{2}\right]}$$

$$\boxed{\omega = \sqrt{\frac{2\left(m_1 g H - m_2 g HR_2/R_1\right)}{\left[m_1R_1^2 + m_2R_2^2 + \frac{m_3R_1^2}{2}\right]}}}$$

ANS

Impulse - Momentum methods for rigid bodies

Recall.

Impulse - momentum theorem

$$\underbrace{\begin{matrix}\text{Total Linear}\\ \text{momentum}\\ \text{final}\end{matrix} - \begin{matrix}\text{Total linear}\\ \text{momentum}\\ \text{initial}\end{matrix}}_{\text{of the whole system}} = \underbrace{\int_{t_i}^{t_f} \left(\Sigma \bar{F}\right) dt}_{\text{impulse of all external forces on the system}}$$

TRUE FOR SYSTEMS WITH PARTICLES RIGID BODIES ETC.

impulse $= 0$ and linear momentum is conserved

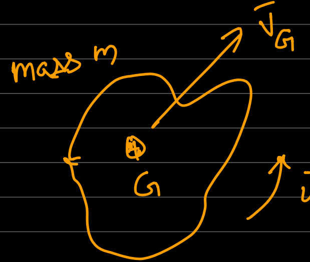
when

(1) $\left(\Sigma \bar{F}\right) = \bar{0}$. (no net forces)

(or) (2) $\Sigma \bar{F}$ is finite. eg. gravity, springs, etc

& $t_f \to t_i$, so that the integral $\to 0$

Linear momentum for a single rigid body



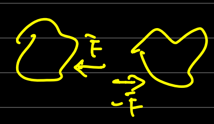
$$\left.\begin{matrix}\text{Linear}\\ \text{Momentum}\end{matrix}\right] = m \bar{v}_G$$

For multiple rigid bodies,

$$\left.\begin{matrix}\text{total linear}\\ \text{momentum}\end{matrix}\right] = m_1 \bar{v}_{G_1} + m_2 \bar{v}_{G_2} + \cdots$$

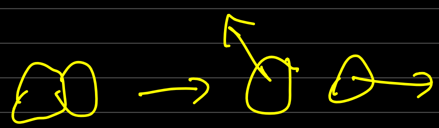
For the purposes of linear momentum conservation "infinite forces"

- are forces of interaction in a collision that happens over infinitesimal duration. $(t_f \to t_i)$

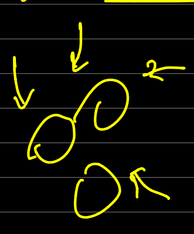


$\bar{F}$ & $-\bar{F} \to \infty$ as $t_f - t_i \to 0$ while the objects still have their velocities change due to a collision.

- analogous interaction forces during "explosions" in which the component pieces experience a finite change in velocity in infinitesimal time.

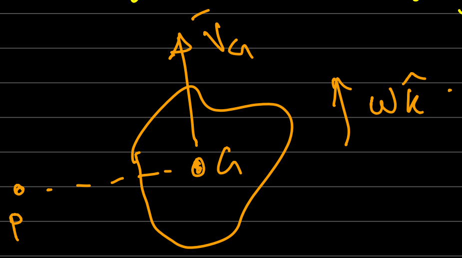


Angular momentum & angular impulse



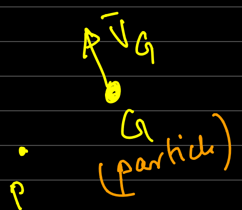
$$\left.\begin{array}{l}\text{angular}\\ \text{impulse}\\ \text{about } O\end{array}\right\} = \int_{t_i}^{t_f} \left(\Sigma M\right)_{/O} \, dt$$

Angular momentum about G of a rigid body $= I_G \omega \hat{k}$ in 2D.



Angular momentum about a different point P for a single rigid body. $= I_G \omega \hat{k} + \bar{r}_{G/P} \times m\bar{v}_G$

Contrast this with angular momentum for a particle G about point P $= \bar{r}_{G/P} \times m\bar{v}_G$.



Change in angular momentum $=$ angular impulse.

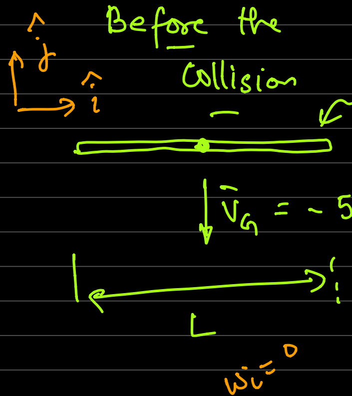
- angular impulse $= 0$ if
    - if $(\Sigma \bar{M})_P$ external $= 0$
        - due to no forces on the object
        - forces all pass through P (central forces).
        - net moment $= 0$ due to various moments canceling.
    - $\Sigma M =$ finite but $t_f - t_i \to 0$ so that $\int_{t_i}^{t_f} (\Sigma M)\, dt \to 0$.

Example : collision, angular momentum conservations.

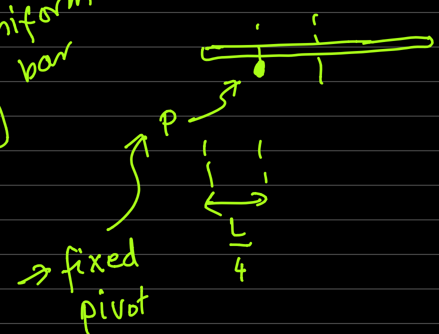
Say collision happens in infinitesimal time.

Before the Collision

thin pencil / uniform bar

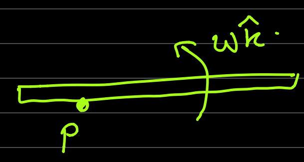


Collision with P



After collision

the pencil rotates about pivot P without slip.

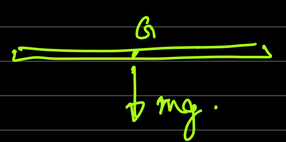


Question: Find ω after the collision.

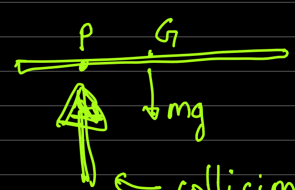
Note! pivot P is fixed and cannot move. [its mass is not known — effectively infinite as it can't be moved]

FBD

Before collision



During the collision



Angular momentum of bar conserved about point P because

- the infinite force acts through P and has no moment about P

- mg has a finite moment about P but when integrated over infinitesimal duration

$$\int (\Sigma M)\, dt \to 0$$

Angular momentum of pencil before collision about P = angular momentum after collision about P

$$\bar{H}_{/P\,i} = \bar{H}_{/P\,f}.$$

$$\bar{H}_{/P\,i} = I_G \omega_i \hat{k} + \bar{r}_{G/P} \times m\bar{v}_{G\,i}$$

$$= 0\hat{k} + \frac{L}{4}\hat{i} \times m(-4\hat{j})$$

$$= \frac{mL}{4}(-4)\hat{k} = -mL\hat{k}.$$

using parallel axis theorem

METHOD 1

$$\bar{H}_{/P\,f} = I_G \omega_f \hat{k} + \bar{r}_{G/P} \times m\bar{v}_{G\,f}$$

$$= \frac{mL^2}{12}\omega_f \hat{k} + \left(\frac{L}{4}\hat{i}\right) \times m\left(\frac{L}{4}\omega_f \hat{j}\right)$$

$$= \frac{mL^2}{12}\omega_f \hat{k} + \hat{k}\left[\frac{mL^2\omega_f}{16}\right]$$

$$= mL^2\omega_f\left[\frac{1}{12} + \frac{1}{16}\right]\hat{k}$$

METHOD 2

$$= I_P \omega_f \hat{k}.$$

$$= \omega_f \hat{k}\left(I_G + m\left(\frac{L}{4}\right)^2\right)$$

$|GP|^2$

$$= \hat{k}\, mL^2\left[\frac{1}{12} + \frac{1}{16}\right]$$

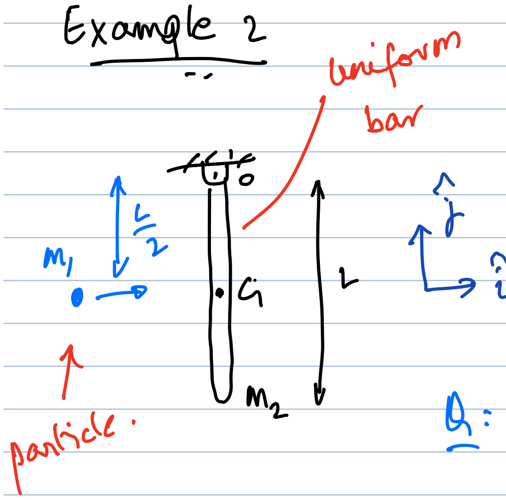
$$-\cancel{m}L\,\hat{k} = \cancel{m}L^2\omega_f\left[\frac{1}{12}+\frac{1}{16}\right]\hat{k}$$

$$\omega_f = -\frac{L}{L^2}\left(\frac{1}{\frac{1}{12}+\frac{1}{16}}\right)$$

given $L = 0.1\,m.$ J

$$\boxed{\omega_f = -\frac{1}{0.1}\cdot\frac{1}{\frac{1}{12}+\frac{1}{16}}}\ \text{rad/s.}$$

Example 2



Initial velocity of mass m_1 = v. ← horizontal. ← just before collision

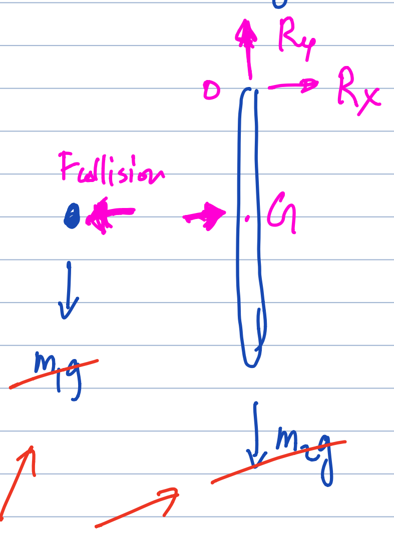
Initial linear + ang. velocity of pendulum = 0. (rest)

<u>Q</u>: what is the angular velocity of the pendulum.

given that the collision is <u>plastic</u> or "<u>perfectly inelastic</u>". ≡ [this means that after the collision the mass m_1 <u>sticks</u> to the pendulum and moves with the pendulum]

<u>Solution</u>

FBD during collision



ignore during collision as they are small compared to collision forces

Big forces ("infinity")

- $F_{collision}$
- R_x and R_y reaction forces ≡ <u>could also</u> be infinity.

because they prevent O from moving despite the large collision forces.

Which point should we take "angular momentum" ($\bar{H}$) conservation about?

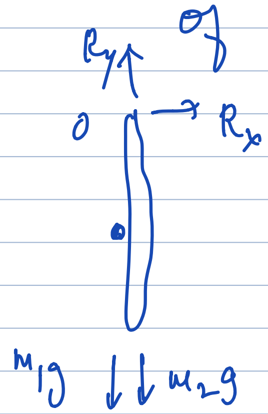
$$\bar{H}_f - \bar{H}_i = \underset{\text{angular impulse}}{\int \Sigma M \, dt}$$

If we take $\bar{H}$ about G, there would be an angular impulse due to the BIG reaction forces $\bar{R}_x$, $\bar{R}_y$. ← which we do not want.

If we take $\bar{H}$ about point O,
for the whole system (pendulum + mass)
then (1) R_x R_y will have an effect because they act at O.

(2) $F_{collision}$ is an internal force of the whole system and does not appear in an FBD of the whole system

true whether elastic or plastic



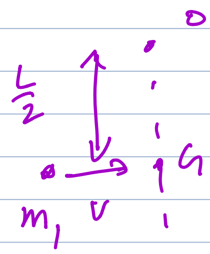
$$\bar{H}_{/O\,i} = \bar{H}_{/O\,f}$$

because no large moments

about O

(note: this applies only if we consider the system and not the pendulum alone).

$$\bar{H}_{/O_i} = \bar{I}_{G_1}\omega_i\hat{k} + \bar{r}_{G_1/O} \times m_1\bar{v}_{i_1}\Big]\ \text{particle}$$
$$+ \bar{I}_{G_2}\omega_i\hat{k} + \bar{r}_{G_2/O} \times m_2\bar{v}_{i_2}\Big]\ \text{pendulum}$$



$\bar{v}_i = v\hat{i}$

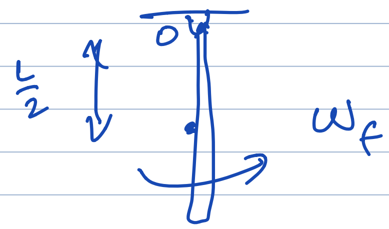
$\bar{r}_{G/O} = -\frac{L}{2}\hat{j}$.

$$= \bar{0} + -\frac{L}{2}\hat{j} \times m_1 v\hat{i}$$
$$+ \bar{0} + \bar{0}$$

entirely due to particle.

$$= + m_1 v \frac{L}{2}\hat{k}.$$

$$\bar{H}_{/O_f} = \text{particle} + \text{pendulum}.$$



$$= \underbrace{\bar{I}_{O_1}\omega_f}_{\text{particle}}\hat{k} + \underbrace{\bar{I}_{O_2}\omega_f\hat{k}}_{\text{pendulum}}.$$

$$\bar{I}_{O_2}\ (\text{pendulum}) = \bar{I}_{G_2} + m_2\left(\frac{L}{2}\right)^2$$ parallel axis thm.

$$= \frac{m_2L^2}{12} + \frac{m_2L^2}{2} = \frac{m_2L^2}{3}.$$

$$\bar{I}_{O_1}\ (\text{particle}) = \cancel{\bar{I}_{G_1}}^{\,0} + m_1\left(\frac{L}{2}\right)^2$$ because $I_G = 0$ for particles

$$= m_1 L^2/4$$

$$\bar{H}_{/o\,f} = \left(I_{o1} + I_{o2}\right)\omega_f\,\hat{k}$$

$$= \left(\frac{m_2 L^2}{3} + \frac{m_1 L^2}{4}\right)\omega_f\,\hat{k}.$$

$$\bar{H}_{/o\,i} = \bar{H}_{/o\,f}$$

$$m_1 v \frac{L}{2} = \left(\frac{m_2 L^2}{3} + \frac{m_1 L^2}{4}\right)\omega_f$$

So $$\boxed{\omega_f = \frac{m_1 v L/2}{\dfrac{m_2 L^2}{3} + \dfrac{m_1 L^2}{4}}}.$$ ANS

<u>what if the collision was elastic?</u>

elastic $\Rightarrow$ KE conserved.

m_1 will have a v_{f1} unknown, and not be <u>stuck</u> to the pendulum

2 unknowns: ω_f pendulum

v_{f1} - mass m_1.

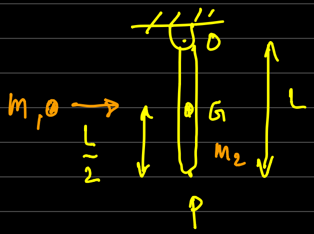
2 eqns: (1) angular momentum conserved abt o

(2) KE conserved.

Solve for ω_f & v_{f1}.

Example

A rigid body pendulum (uniform bar) of length L and mass m_2 hangs from O as shown.

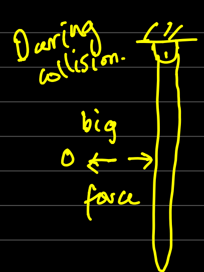


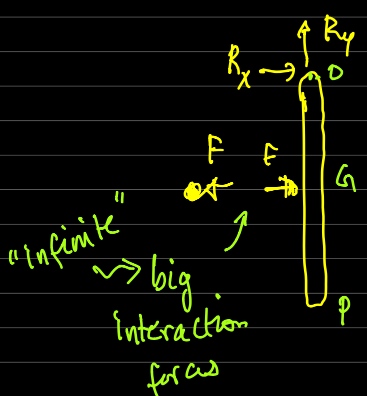
Another point mass (particle) m_1 collides with the pendulum, exactly hitting G on the pendulum. It is plastic collision, which means that the 2 masses are stuck together / move together after the collision.

m_1 is traveling horizontally with speed v just before the collision.

Q: What is the angular velocity of the pendulum + m_1 just after the collision?

Solution

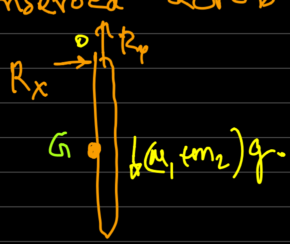




← reaction forces at O can also be "infinite"

we ignore finite forces like gravity.

Angular momentum of the WHOLE SYSTEM is conserved across the collision about point O.



because the only potentially large forces pass through O.

$$\bar{H}_{/o_i} = ? = \left(\bar{H}_{/o_i} \text{ of } m_1\right) + \left(\bar{H}_{/o_i} \text{ of } m_2\right)$$

$$= \bar{r}_{G/o} \times m_1 \underbrace{(v\,\hat{i})}_{\text{velocity of } m_1 \text{ before collision}} + \bar{0}$$

$\bar{0}$ ← pendulum initially at rest

$$= \left(-\frac{L}{2}\hat{j}\right) \times m_1(v\,\hat{i}) = \frac{m_1 v L}{2}\left(-\hat{j}\times\hat{i}\right)$$

$$= \frac{m_1 v L}{2}\hat{k}.$$

① $\boxed{\bar{H}_{/o_i} = \frac{m_1 v L}{2}\hat{k}.}$

$$\bar{H}_{/o_f} = \underbrace{\bar{r}_{G/o} \times m_1 \bar{V}_{1_f}}_{\text{of mass } m_1} + \underbrace{I_o\,\omega_f\,\hat{k}}_{\text{because pendulum rotates with center of rotation } O.} \quad ②$$

$$\bar{V}_{1f} = \frac{L}{2}\omega_f\,\hat{i} \quad ③$$

$$I_o = I_G + m_2\underbrace{\left(\frac{L}{2}\right)^2}_{|OG|^2} = \frac{m_2 L^2}{12} + \frac{m_2 L^2}{4} = \frac{m_2 L^2}{3}. \quad ④$$

Use ③ & ④ in ②

$$\bar{H}_{/o_f} = \left(-\frac{L}{2}\hat{j}\right) \times \left(m_1\frac{L}{2}\omega_f\,\hat{i}\right) + \frac{m_2 L^2}{3}\omega_f\,\hat{k}$$

$$= m_1\frac{L^2}{4}\omega_f\,\hat{k} + \omega_f m_2\frac{L^2}{3}\hat{k} = L^2\left(\frac{m_1}{4} + \frac{m_2}{3}\right)\omega_f\,\hat{k} \quad ⑤$$

$$\bar{H}_{/o_i} = \bar{H}_{/o_f}$$

$$(1) = (5)$$

$$\frac{m_1 v \cancel{L}}{2} \cancel{\hat{k}} = L^{\cancel{2}} \left(\frac{m_1}{4} + \frac{m_2}{3} \right) \omega_f \cancel{\hat{k}}$$

ANS

$$\boxed{\omega_f = \frac{m_1 v}{2L\left(\frac{m_1}{4} + \frac{m_2}{3} \right)}}$$

final angular velocity of the whole system.

Vibrations

- a type of dynamic behavior in which the system oscillates about an equilibrium point

Why study vibrations?

- causes discomfort
- acts as an excitation for sound and noise, can be "good" or "bad"
- results in large stresses and catastrophic failures

Wind induced vibrations
Search for: Tacoma narrows bridge, July-Nov 1940.

Resonance

Search for: resonance, helicopter

Double pendulum

Search for: strogatz, double pendulum on youtube

An application from a few years ago.

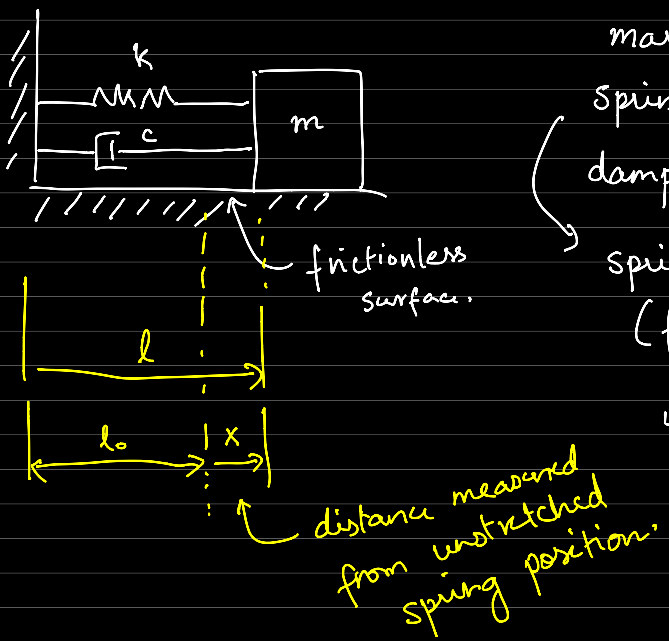
Eavesdropping using video

Search for: the visual microphone

Many degrees of freedom system

search for: ‘the resonant bridge’ on youtube
Messiah College, PA

MASS- SPRING- DAMPER SYSTEM



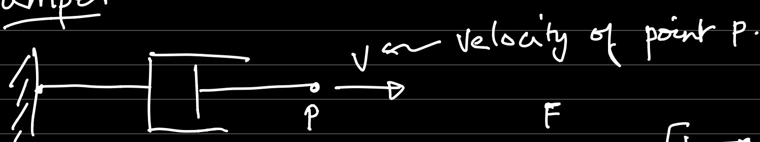
mass = m

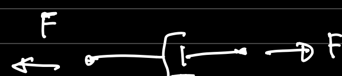
spring stiffness = k

damping coefficient = c

spring rest length = l_0
(force-free length
stress-free length
unstretched length)

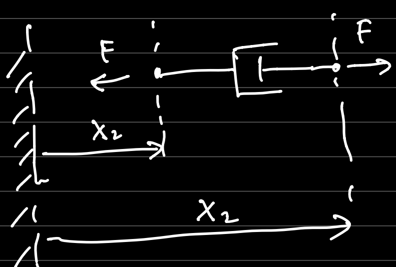
Damper





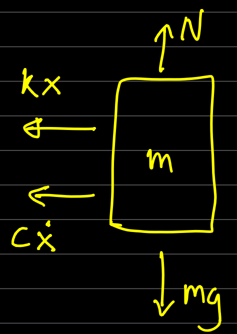
$F = +cV$

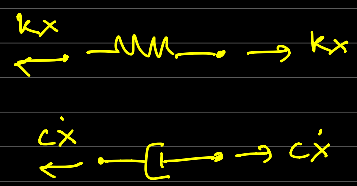
Same as "linear" drag"



Then, $F = +c(\dot{x}_2 - \dot{x}_1)$.

Eg dampers in door closing mechanisms.





$$\Sigma F_x = m a_x$$

$$-kx - c\dot{x} = m\ddot{l}$$

$$-kx - c\dot{x} = m\ddot{x}$$

because $l = l_o + x$

$\Rightarrow \ddot{l} = \ddot{x}$

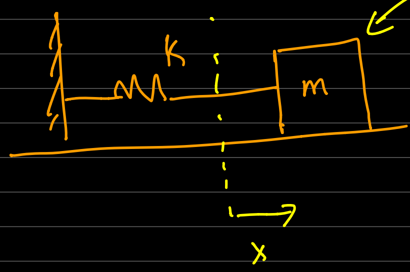
as l_o = constant.

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Equation of motion for a mass-spring-damper system.

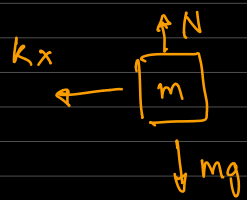
2^{nd} order linear ordinary differential equation

FREE VIBRATION . WITH NO DAMPING (SIMPLE HARMONIC OSCILLATOR)



Mass-spring system

x is measured from the rest length of spring (free length, stress-free length, etc.)



$$m a_x = -kx$$

$$m\ddot{x} = -kx \Rightarrow \boxed{m\ddot{x} + kx = 0} \quad (1)$$

We would like to derive an expression for $x(t)$ that describes the free vibration of mass m.

We know that the motion will be an oscillation, which will persist for ever.

Let us try $x(t) = A\cos(\omega t) + B\sin(\omega t)$ (2)

ω – allows for different frequencies

A, B – allows for different amplitudes

$$\dot{x} = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$\ddot{x} = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \quad - (3)$$

Plug (3) and (2) in (1)

$$m\ddot{x} + kx = 0$$

$$m\left(-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right) + k\left(A\cos\omega t + B\sin\omega t\right) = 0$$

$$-m\omega^2\left(A\cos\omega t + B\sin\omega t\right) + k\left(A\cos\omega t + B\sin\omega t\right) = 0$$

$$\left(-m\omega^2 + k\right)\left(\cancel{A\cos\omega t + B\sin\omega t}\right) = 0.$$

$$\Rightarrow \boxed{k - m\omega^2 = 0}$$

because $A\cos\omega t + B\sin\omega t = 0$ means $x(t) = 0$, "trivial solution".

$$\omega^2 = \frac{k}{m} \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

4 $\boxed{\omega = \sqrt{\frac{k}{m}}}$. ω is called the "angular frequency" of oscillation.

units. s^{-1}, Hz, rad/s,

- ω increases with increasing k.
- ω decreases with increasing m.

$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$ n - denotes "natural frequency" of the system.

A and B are determined from knowledge of initial conditions

Say x_0 is the initial value of x. $= x(0)$

$v_0 = \dot{x}_0$ is the initial value of $\dot{x}$. $= \dot{x}(0)$.

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$x(0) = x_0$$

$$A\cos(0) + B\sin(0) = x_0$$

$$A \cdot 1 + B \cdot 0 = x_0$$

(5) $$\boxed{A = x_0}$$

$$\dot{x}(0) = \dot{x}_0$$

$$\dot{x}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$\dot{x}(0) = -A\omega \cdot 0 + B\omega \cdot 1 = B\omega = \dot{x}_0$$

$$\Rightarrow B\omega = \dot{x}_0$$

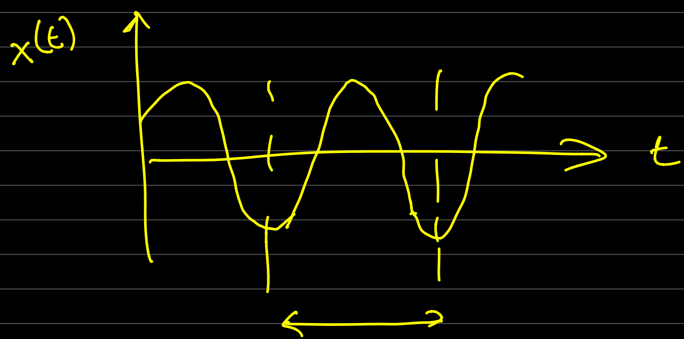
(6) $$\boxed{B = \frac{\dot{x}_0}{\omega}}$$

Plug (5), (6) in eqn (2)

(7) $$\boxed{x(t) = x_0\cos(\omega_n t) + \frac{\dot{x}_0}{\omega} \cdot \sin(\omega_n t)}$$

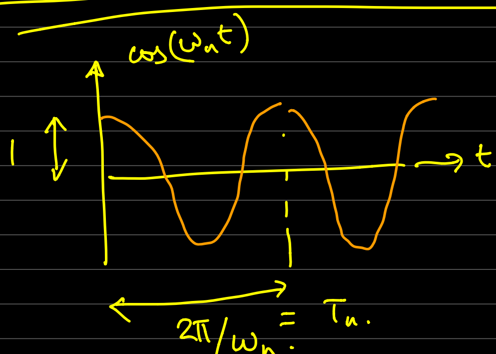
where $\omega_n = \sqrt{\frac{k}{m}}$

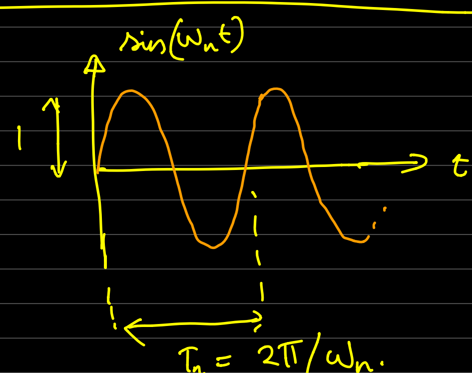
simple harmonic motion



$T_n = \frac{2\pi}{\omega_n}$

What does $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$ look like?





T_n = time period of oscillation of free vibration $= \frac{2\pi}{\omega_n}$ = Time period for one full cycle after which everything repeats

= natural period. (units: seconds).

$f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi}$ = natural frequency (units: $\frac{1}{s}$, Hz).

→ how many cycles per second.

ω_n = "natural angular frequency"

Getting more math. intuitions about $x(t)$

Lets go back to $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$.

$$x(t) = \sqrt{A^2+B^2} \cdot \left[\frac{A}{\sqrt{A^2+B^2}} \cos(\omega_n t) + \frac{B}{\sqrt{A^2+B^2}} \cdot \sin(\omega_n t) \right]$$

$$C_1 = \frac{A}{\sqrt{A^2+B^2}}, \quad C_2 = \frac{B}{\sqrt{A^2+B^2}}.$$

$$C_1^2 + C_2^2 = \frac{A^2}{A^2+B^2} + \frac{B^2}{A^2+B^2} = \frac{A^2+B^2}{A^2+B^2} = 1.$$

Say, $C_1 = \cos(\phi)$, $C_2 = \sin(\phi)$, where ϕ is some angle.

$$x(t) = \sqrt{A^2+B^2} \left[\cos(\omega_n t) \cos\phi + \sin(\omega_n t) \cdot \sin\phi \right]$$

Trig. identity: $\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 = \cos(\theta_1 - \theta_2)$.

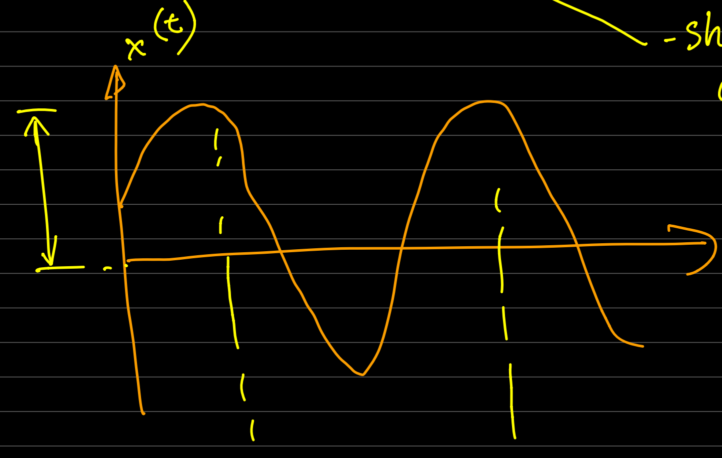
$$\boxed{x(t) = \sqrt{A^2 + B^2} \cdot \cos(\omega_n t - \phi)}$$

amplitude

phase offset.

$x(t)$ is a cosine

time-shifted cosine



$$\sqrt{A^2 + B^2} = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega^2}}$$

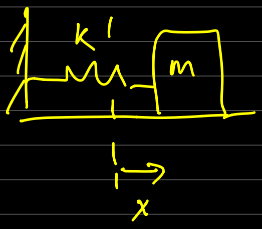
$$T_n = \frac{2\pi}{\omega_n}$$

$$\frac{\phi}{2\pi}$$

where the peak occurs

Example

Mass - spring system with no damping



$m\ddot{x} = -kx$. Given mass $m = 2$ kg, spring stiffness $k = 4$ N/m.

$$\boxed{m\ddot{x} + kx = 0}$$

initial $x(0) = 1$ m , $\dot{x}(0) = 2$ m/s.

Determine

(1) natural frequency ω_n of the system (angular frequency)

(2) the time period of free vibrations.

(3) $x(t)$ given the above initial conditions.

---

(1) $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{2}} = \sqrt{2} \text{ rad/s}$$, (or) $\sqrt{2}$ Hz.

ANS

(2) time period of oscillation. $$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2} \text{ seconds}$$

(3) $$x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} \sin(\omega_n t)$$

$$\boxed{x(t) = 1 \cos(\sqrt{2}\, t) + \frac{2}{\sqrt{2}} \sin(\sqrt{2}\, t)}$$

ANS

$$x(t) = A\cos(\sqrt{2}\,t) + B\sin(\sqrt{2}\,t).$$

$$x(0) = 1.$$

$$x(0) = A\,.\cos(0) + B\sin(0) = A\,. = 1.$$

$$\boxed{A = 1}\ m$$

$$\dot{x}(0) = 2$$

$$\dot{x}(t) = -\sqrt{2}A\sin(\sqrt{2}\,t) + B\sqrt{2}\cos(\sqrt{2}\,t).$$

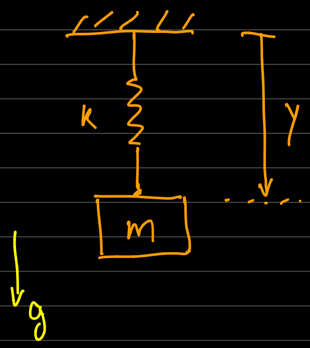
$$\dot{x}(0) = 2 = -\sqrt{2}A\,.(0) + B\sqrt{2}\,.(1)$$

$$B = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\boxed{B = \sqrt{2}}\ m.$$

$$\boxed{x(t) = 1.\cos(\sqrt{2}\,t) + \sqrt{2}\sin(\sqrt{2}\,t)}$$ ANS

Example Vertical oscillations of a mass hanging down



Mass $m = 2$ kg

Spring stiffness $k = 4$ N/m

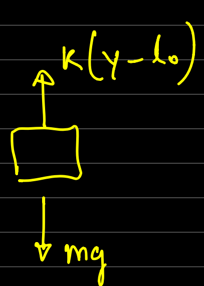
rest / unstretched / force-free length of spring $= l_0 = 1$ m

gravity g.

a) Natural frequency ω_n of the system (angular freq.)

b) Natural frequency f_n of the system.

c) Motion $y(t)$ of the system, given $y(0) = 2$ m and $\dot{y}(0) = 1$ m/s.



where $(y - l_0)$ is spring stretch from its unstretched length.

$$\Sigma F_y = m a_y$$

$$\boxed{mg - k(y - l_0) = m\ddot{y}}$$

$$\boxed{m\ddot{y} + ky = mg + kl_0} \quad \text{--- (1)}$$

Can we convert this equation to the standard form $\boxed{m\ddot{x} + kx = 0}$? (2)

Static Equilibrium value of y from (1).

What length y will the system be in static equilibrium?

Set $\ddot{y} = 0$ (static equilibrium $\Rightarrow \ddot{y} = 0$).

in (1)

$$\cancel{m\ddot{y}} + ky = kl_0 + mg.$$

$$y = \frac{kl_0 + mg}{k}$$

$$(3) \quad \boxed{y_{st} \text{ or } \delta_{st} = l_0 + \frac{mg}{k}}$$

static equilibrium value of y.

Define a new quantity

$$x = y - \delta_{st}$$

$$(4) \quad x = y - \left(l_0 + \frac{mg}{k}\right)$$

x is the deviation from the static equilibrium position.

(5) Differentiate (4).

$$\ddot{x} = \ddot{y}$$

Use (4) and (5) in (1)

$$m\ddot{y} + ky = kl_0 + mg$$

$$m\ddot{x} + k\left(x + l_0 + \frac{mg}{k}\right) = kl_0 + mg$$

$$m\ddot{x} + kx + \cancel{kl_0} + k \cdot \frac{\cancel{mg}}{k} = \cancel{kl_0} + \cancel{mg}$$

$$\boxed{m\ddot{x} + kx = 0}$$

Therefore, the system has the same behavior as mass oscillitating horizontally

except we need to use $y = x + l_0 + \frac{mg}{k}$.

Solve for $x(t)$.

$$y(t) = x(t) + l_0 + \frac{mg}{k}$$

More generally,

(1) find static equilibrium value δ_{st}

by setting derivative terms to zero

(2) $y(t) = \delta_{st} + x(t)$

(a) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{2}} = \sqrt{2}$ Hz (angular freq)

(b) $f_n = \frac{\omega_n}{2\pi} = \frac{\sqrt{2}}{2\pi} = \frac{1}{\sqrt{2}\cdot\pi}$ Hz.

(c) $y(t) = x(t) + \delta_{st}$

$= x(t) + l_0 + \frac{mg}{k}$

$\delta_{st} = l_0 + \frac{mg}{k} = 1 + \frac{2 \times (9.81)}{4} = 1 + \frac{9.81}{2}$

$= 5.905$ m.

$x(t) = ?$

$x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} \sin(\omega_n t).$

$y_0 = 2$ m. $\dot{y}_0 = 1$ m/s.

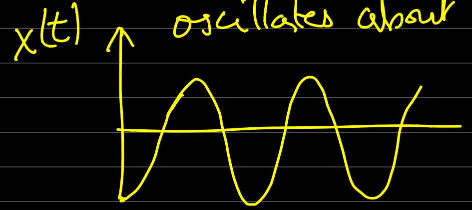
$x_0 = y_0 - \delta_{st} = 2 - 5.905 = -3.905$ m.

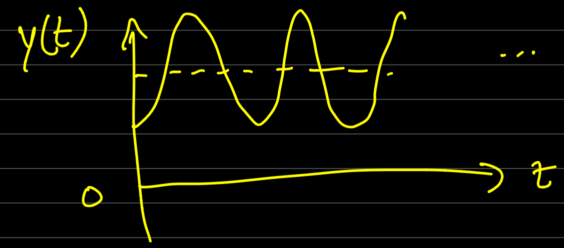
$\dot{x}_0 = \dot{y}_0 = 1 \ m/s.$

$x(t) = -3.905 \cos(\sqrt{2}\,t) + 1.\sin(\sqrt{2}\,t)$

$$y(t) = 5.905 - 3.905 \cos(\sqrt{2}\,t) + 1.\sin(\sqrt{2}\,t)$$

$x(t)$ oscillates about $x=0$



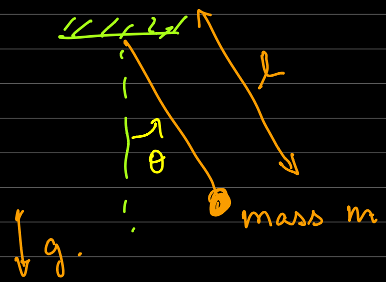


$\ldots\ 5.905 = \delta_{st}$

$$\left.\begin{array}{l}\text{Amplitude of}\\ \text{oscillation}\end{array}\right] = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$$

$$= \sqrt{3.905^2 + 1^2} \quad m$$

Example SIMPLE PENDULUM OSCILLATIONS.



$$ml^2\alpha = -mgl\sin\theta \quad (1)$$

where α = angular acceleration $= \ddot{\theta}$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Divide by ml^2

$$\ddot{\theta} + \frac{mgl}{ml^2}\sin\theta = 0$$

(2) $$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

↳ similar to $m\ddot{x} + kx = 0$ — $\ddot{x} + \frac{k}{m}x = 0$

"Small angle approximation" or "linearization"

for small θ, we have $\sin\theta \approx \theta$.

(or) the Taylor series of $\sin\theta$ about $\theta = 0$ has θ as the first term.

Replace $\sin\theta$ with θ.

(3) $$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$ ← linearized equation

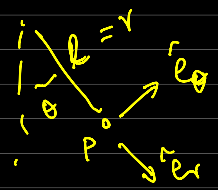
Exact correspondence with $m\ddot{x} + kx = 0$ (or) $\ddot{x} + \frac{k}{m}x = 0$

For (3)

$$\boxed{\omega_n = \sqrt{g/l}}$$

$$f_n = \frac{\omega_n}{2\pi}, \quad T_n = \frac{2\pi}{\omega_n} \quad \text{etc.}$$

Recap Deriving simple pendulum equations



polar coordinates (motion in a circle)

$$\bar{a}_p = -r\dot{\theta}^2\,\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta$$

$$\Sigma\bar{F} = m\bar{a}_p$$

in $\hat{e}_\theta$ direction: $-mg\sin\theta = mr\ddot{\theta}$

$$= ml\ddot{\theta}$$

$$ml\ddot{\theta} + mg\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

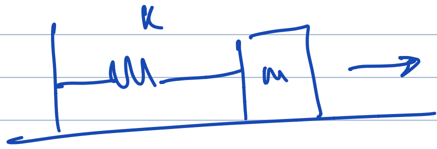
Forced vibration : mass-spring system. No damping

(1) Free : $m\ddot{x} + kx = 0$

(2) Forced : $m\ddot{x} + kx = \underbrace{F_0 \cos \omega t}$ — external oscillatory force.

ω — forcing frequency

F_0 — amplitude of external force.



$\longrightarrow F(t) = F_0 \cos(\omega t)$.

Note: ω may or not be equal to ω_n.

In math, they'd call (1) as a "homogeneous equation".
An (2) is called "inhomogeneous" equation
find a "particular solution" corresponding to
the RHS $F_0 \cos(\omega t)$.

↳ Something when plugged into (2) satisfies (2).

Try out a particular solution of the form:

$$x_p(t) = D \cos(\omega t) . \quad (3)$$

Use (3) in (2).

$$\dot{x}_p(t) = -\omega D \sin \omega t$$

$$\ddot{x}_p(t) = -\omega^2 D \cos \omega t .$$

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

$$m\left(-\omega^2 D\cos(\omega t)\right) + k\left(D\cos(\omega t)\right) = F_0\cos(\omega t)$$

$$D\,\cancel{\cos(\omega t)}\left[-m\omega^2 + k\right] = F_0\,\cancel{\cos(\omega t)}$$

$$\boxed{D = \frac{F_0}{k - m\omega^2}}$$

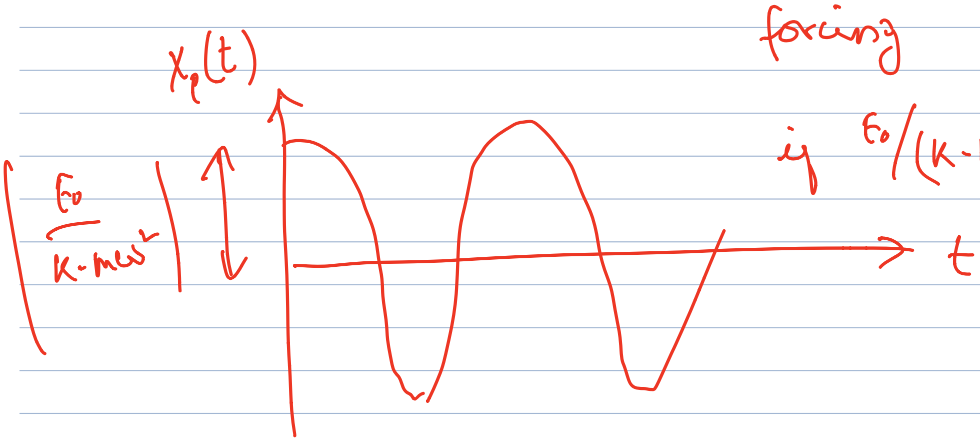
So, a particular solution $x_p(t)$ when $F(t) = F_0\cos\omega t$

is $x_p(t) = D\cos(\omega t)$

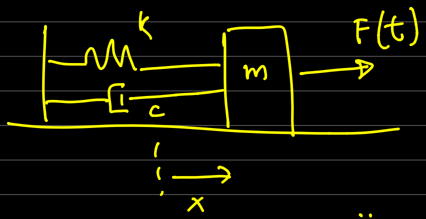
$$\boxed{x_p(t) = \underbrace{\frac{F_0}{k - m\omega^2}} \cdot \cos(\omega t).}$$

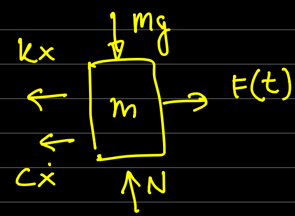
amplitude of oscillation in response to the external forcing ← forced vibration response amplitude.

if $F_0/(k - m\omega^2) > 0$



FORCED VIBRATIONS



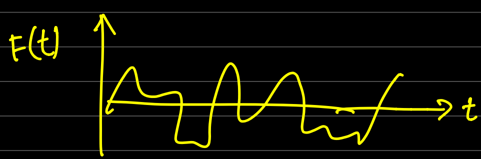


$$m\ddot{x} = -kx - c\dot{x} + F(t)$$

$$\boxed{m\ddot{x} + kx + c\dot{x} = F(t)}$$ Damped forced vibrations

Undamped forced vibration

$$m\ddot{x} + kx = F(t)$$



For simplicity.

$F(t) = F_0 \sin(\omega t)$.

sinusoidal forcing

ω - frequency of the external forcing $F(t)$

F_0 - Amplitude of $F(t)$

$\omega \neq \omega_n$ in general.

$$m\ddot{x} + kx = F_0 \sin \omega t \quad (1)$$

what is "forced response" $x(t)$?

↳ response to $F_0 \sin\omega t$.

"particular solution" for $F_0 \sin \omega t$.

$$x(t) = A \sin(\omega t) \quad (2)$$

Use (2) in (1)

$$\dot{x} = +A\omega\cos(\omega t) \quad \& \quad \ddot{x} = -A\omega^2\sin(\omega t).$$

$$m\ddot{x} + kx = F_0 \sin \omega t$$

$$m\left(-A\omega^2 \sin(\omega t)\right) + k\left(A\sin(\omega t)\right) = F_0 \sin \omega t$$

$$A\left(-m\omega^2 + k\right) \cancel{\sin \omega t} = F_0 \cancel{\sin \omega t}$$

$$\left.\begin{array}{c}\text{Amplitude of}\\ \text{force response}\\ x(t)\end{array}\right\} = \boxed{A = \frac{F_0}{k - m\omega^2}} . \quad (3)$$

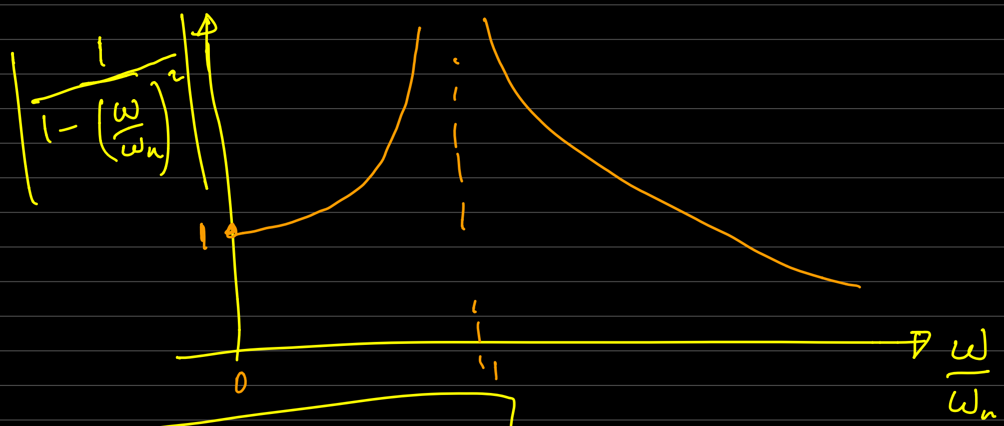
$$A = \frac{F_0/k}{k/k - \frac{m\omega^2}{k}} = \frac{F_0/k}{1 - \frac{\omega^2}{k/m}}$$

$$\boxed{A = \frac{F_0/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2}} \quad (4).$$

Equation (4): Intuition

$$A = \frac{F_0/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2}.$$

$F_0/k = \delta_{st}$ for a given constant force F_0 on the spring of stiffness k.

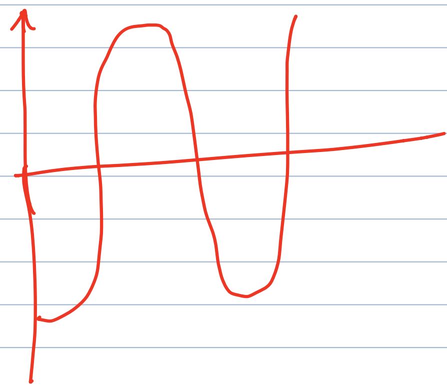


$\frac{\omega}{\omega_n}$	$\left\lvert\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\rvert$	
0	1.	very low frequency forcing
∞	0	very high frequency forcing
1	∞	RESONANCE $\frac{\omega}{\omega_n} = 1$

$\omega = \omega_n$

Forced vibration (continued from previous lecture)

if $F_0/(k - m\omega^2) < 0$ then $x_p(t)$ Respond



RESONANCE (or) More generally, how does response amplitude depend on forcing frequency ω?

$$\text{Response amplitude} = \left| \frac{F_0}{k - m\omega^2} \right|$$

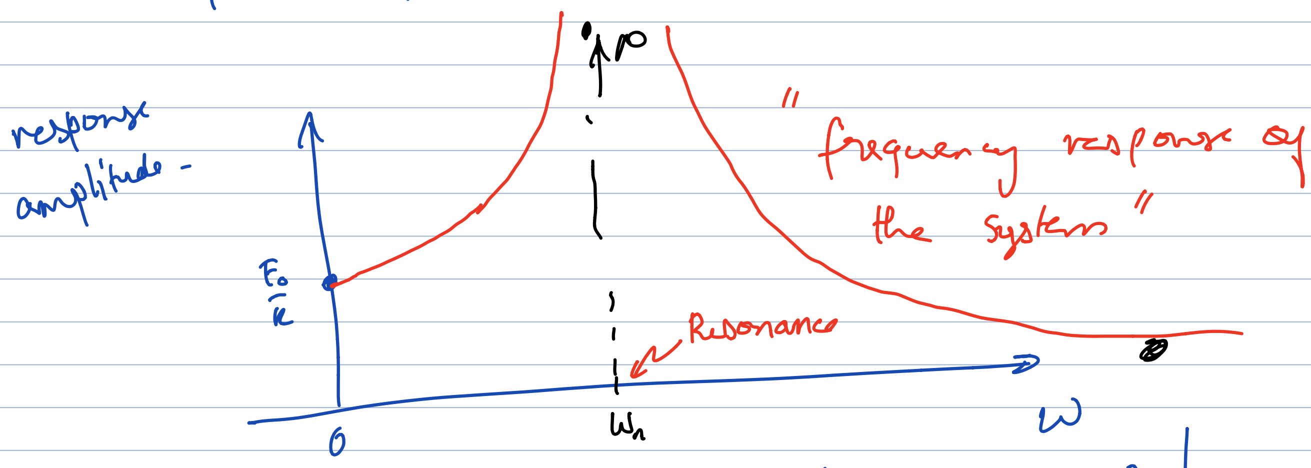
$$= \left| \frac{F_0/k}{(k - m\omega^2)/k} \right| = \left| \frac{F_0/k}{1 - \frac{m\omega^2}{k}} \right|$$

Because $\omega_n = \sqrt{\frac{k}{m}}$. $\omega_n^2 = \frac{k}{m}$. $\Rightarrow$

$$\text{Responce amplitude} = \left| \frac{F_0/k}{1 - \frac{\omega^2}{\omega_n^2}} \right|.$$

F_0, k, m, ω_n - constants of the system

Let's plot response amplitude versus ω.



ω	response amplitude $\left| F_0/k \Big/ \left(1 - \frac{\omega^2}{\omega_n^2}\right) \right|$.
0	F_0/k. $\leftarrow$ statics. $\omega = 0$
ω_n	∞ $\leftarrow$ LARGE OSCILLATIONS.
∞	0

$F(t)$ averages out to 0 when $\omega = \infty$.

- $\omega = \omega_n$, response amplitude $\rightarrow \infty$

"RESONANCE"

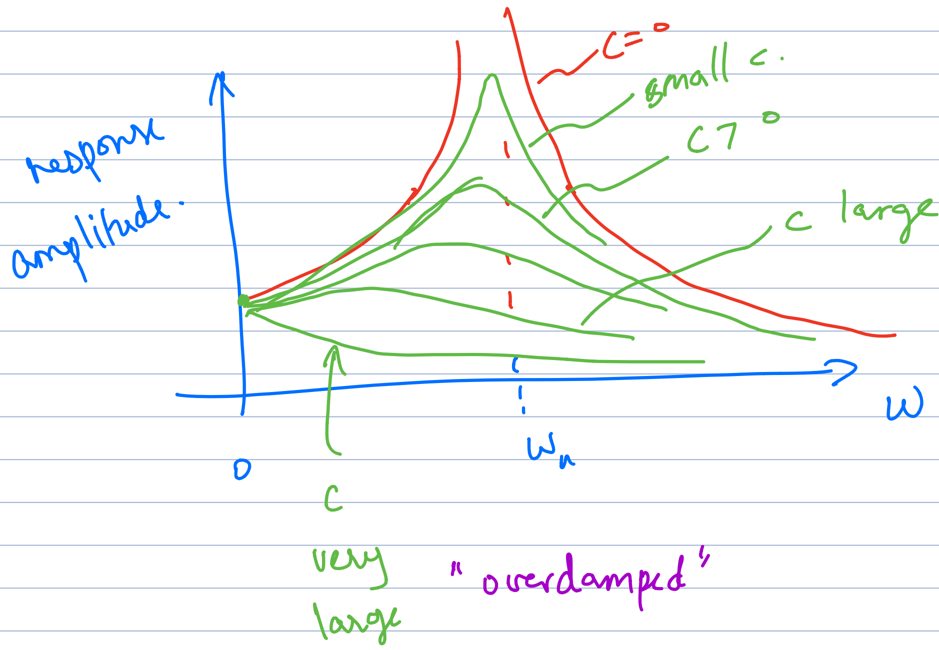
when forcing frequency ω matches or is close to natural frequency ω_n, we get large response amplitudes.

BAD. if we don't like big amplitudes (eg. bridges, buildings, cars).

GOOD if we do want large amplitudes. eg. music

Effect of damping on resonance

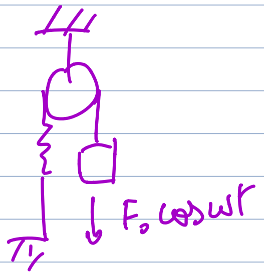
$$m\ddot{x} + kx + c\dot{x} = F_0 \cos \omega t$$



Types of problems in vibrations for 2030

1) Many problems will start with writing "equations of motion".

"$F = ma$" "$\Sigma M = I\alpha$" etc.



→ derive an expression for $\ddot{y}$ or $\ddot{\theta}$ or whatever the key variable is for the problem.

You will get an equation that looks like

$$(G)\,\ddot{y} + (H)\,y = C_0 + F_1 \cos \omega t + F_2 \sin \omega t$$

(no damping)

→ What is the natural frequency of free vibration?

Make analogies to the standard form:

$$m\ddot{x} + kx = F_0 \cos\omega t.$$

$$\omega_n = \sqrt{\frac{k}{m}}.$$

Here: $H \to k$

$G \to m$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{H}{G}}}$$

At what forcing freq,

~~When~~ is resonance?

$$\omega = \omega_n.$$

if $F_1 = 3$ and $F_2 = 7$, determine response amplitude.

$\omega = 2$ Hz.

response amplitude for $m\ddot{x} + kx = F_0 \cos\omega t$

$$\left| \frac{F_0}{k - m\omega^2} \right|.$$

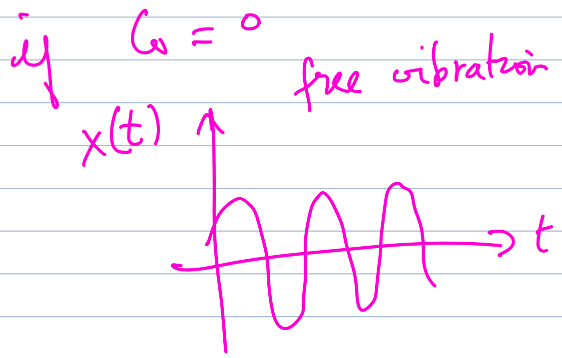
For the new system:

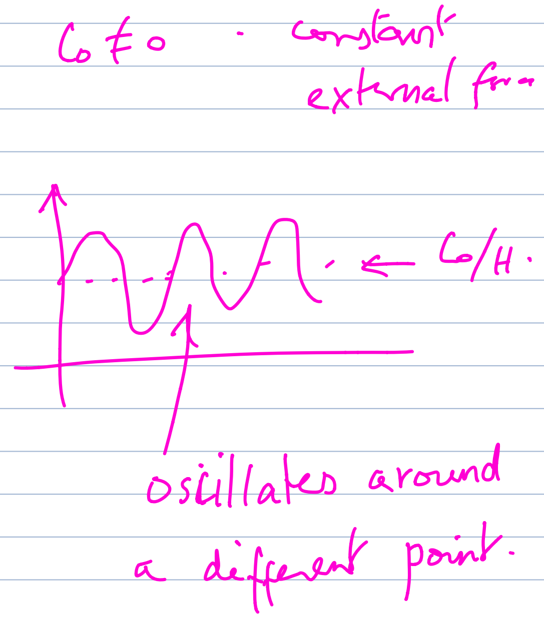
$$D_1 = \left| \frac{F_1}{H - G\omega^2} \right|$$ → in response to $F_1 \cos\omega t$

$$D_2 = \left| \frac{F_2}{H - G\omega^2} \right|$$ in response to $F_2 \sin \omega t$

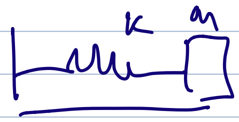
Full response amplitude : $\sqrt{D_1^2 + D_2^2}$. in response to $F_1 \cos \omega t + F_2 \sin \omega t$.

A constant C_0 on the RHS will not affect oscillation amplitude or frequency but just affect the equilibrium point about which free vibration happens.





DONE :

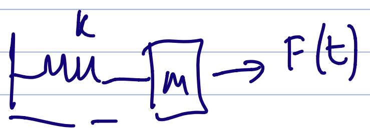
(1) 

$m\ddot{x} + kx = 0$. free vibration
no damping.

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$\omega_n = \sqrt{k/m}.$$

A and B decided by initial conditions $x(0)$, $\dot{x}(0)$.

(2) 

$m\ddot{x} + kx = F_0 \cos\omega t$ (no damping)
forced vibrations

particular solution $x_p(t)$ = response to forcing $= D.\cos(\omega t)$

where $D = \dfrac{F_0}{k - m\omega^2}$. – response amplitude.

and $D \to \infty$ when $\omega \to \omega_n$ "resonance".

Full solution given initial condition

$$x(t) = \underbrace{A\cos\omega_n t + B\sin\omega_n t}_{\text{free vibration}} + \underbrace{\left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t}_{\text{forced vibration}}$$

given initial $x(0)$, $\dot{x}(0)$, can solve for A, B.

③ free vibration with damping.

decaying $x(t)$

$$m\ddot{x} + c\dot{x} + kx = 0 \quad ①$$

solution $x(t)$ of the form $Ae^{\lambda t}$.

(instead of $\cos(\omega t)$ $\sin(\omega t)$).

$$x(t) = Ae^{\lambda t} \quad ②.$$

② in ① : $\dot{x} = \lambda A e^{\lambda t}$. $\ddot{x} = \lambda^2 A e^{\lambda t}$ ③

②, ③ in ①

$$m\left(\lambda^2 \cancel{A} \cancel{e^{\lambda t}}\right) + c\left(\lambda \cancel{A} \cancel{e^{\lambda t}}\right) + k\left(\cancel{A} \cancel{e^{\lambda t}}\right) = 0$$

$$\boxed{m\lambda^2 + c\lambda + k = 0}$$ characteristic equation

quadratic $\lambda = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$ 2 roots. λ_1 and λ_2

Case 1. $c^2 - 4mk > 0$

λ_1, λ_2 real.

solution: $x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

DONE.

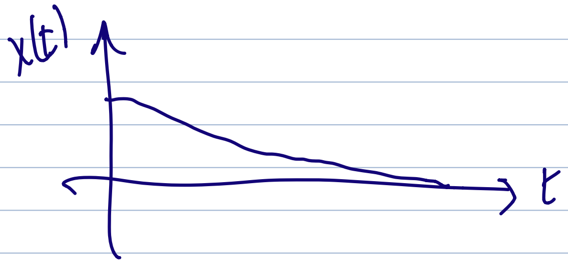
$c^2 - 4mk < 0$

λ_1, λ_2 complex conjugates

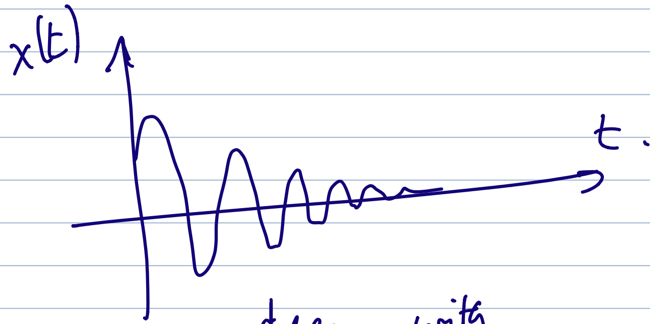
$\lambda_{1,2} = p \pm iq$.

$x(t) = e^{pt}(A\cos qt + B\sin qt)$.

"OVER DAMPED" | "UNDER-DAMPED"



decay with no oscillation

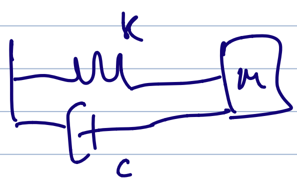


decay with oscillation.

A, B using initial conditions.

"Equations of motion"

Differential equation that describes the system dynamics.

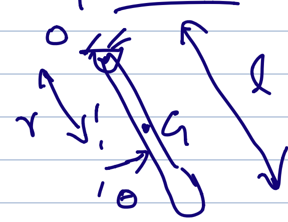


$m\ddot{x} + c\dot{x} + kx = 0$

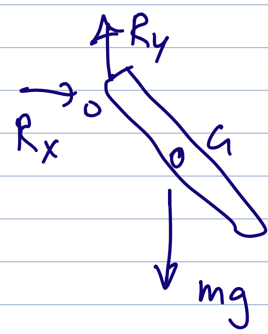
derived using $\Sigma F = ma$

$$\boxed{m\ddot{x} = -kx - c\dot{x}}$$

compound pendulum



I_G.



Take moments about O.

$\Sigma \bar{M}_{/O} = I_{/O}\, \alpha\, \hat{k}$.

$-mgr\sin\theta = (I_G + mr^2)\ddot{\theta}$

$$\boxed{(I + mr^2)\ddot{\theta} + mgr\sin\theta = 0}$$

Differential eqns derived using HW8.

$\Sigma M = I\alpha$, $\Sigma F = ma$

& kinematics eg. rolling without slip

$\sin\theta \approx \theta$ for small θ.

$$\Rightarrow \boxed{(I + mr^2)\ddot{\theta} + mgr\,\theta = 0}$$ ← linearized

$$\omega_n = \sqrt{\frac{mgr}{I + mr^2}}$$

$\theta \leftrightarrow x$

$\ddot{\theta} \leftrightarrow \ddot{x}$

$I + mr^2 \leftrightarrow m$

$mgr \leftrightarrow k$.

etc

use the analogy.

Natural frequency not affected by any RHS.

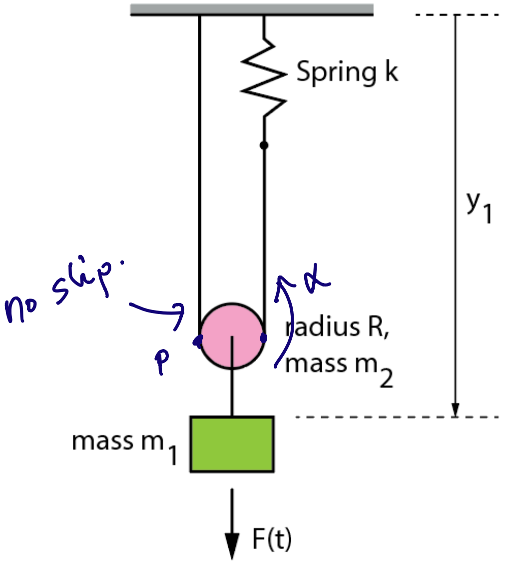
eliminate α, T_2, T_3 to get 1 equation in Y_1

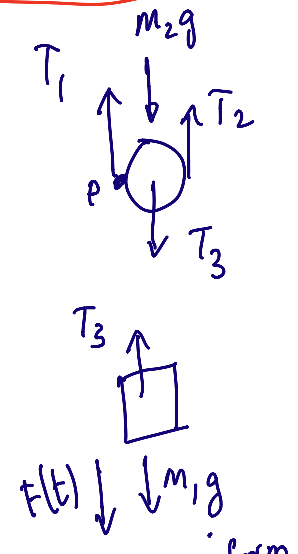
$$\Rightarrow (\quad)\ddot{Y}_1 + (\quad)\Delta Y_1 = \text{const} + ?F(t)$$

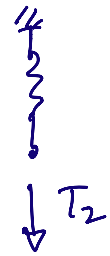
$(y_1 - y_{1_0})$ ← y_1 when spring not stretched.

$m_1 \ddot{y}_1 = m_1 g - T_3$. (1)

Q4. Vibrations [12 points]







$\Sigma M_{/P} = I_P \alpha$.

(2) $T_2(2R) - T_3 R - m_2 g R = \left(\frac{mR^2}{2} + mR^2\right)\alpha$

4 unknowns: $\ddot{y}_1, \alpha, T_2, T_3$.

(3) $\ddot{y}_1 = -R\alpha$ (no slip

uniform disk.

(4) $T_2 = +k(2\Delta y_1)$.

Consider the system shown above, with a rigid body pulley of mass m_2 = 1 kg and radius R = 0.2 m. The mass m_1 = 1 kg. The spring stiffness is k = 100 N/m. When the spring is unstretched, the position of the mass m1 is: $y_1 = 1\,\mathrm{m}$.

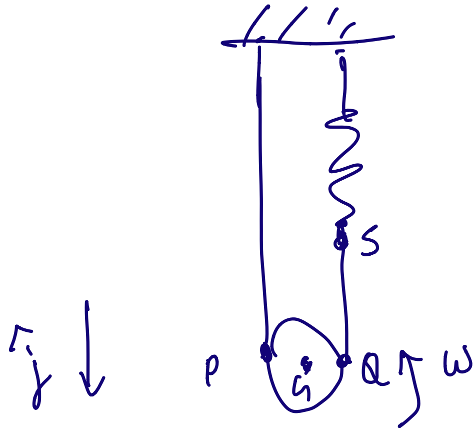
a) **Draw all relevant FBDs.**
b) **Determine the differential "equations of motion" for this system in terms of the vertical position $y_1(t)$ of mass m_1.**
c) What is the **time period of undamped free vibration** of this system if damping c = 0? (free vibration means external force $F(t) = 0$).
d) If the external torque was given by $F(t) = F_0 \cos(\omega t)$ and there was no damping (c = 0), determine the **response amplitude** of oscillation in $y_1(t)$. Given: $\omega = 5\,\mathrm{Hz}, F_0 = 1\,\mathrm{N}$.
e) If external torque $F(t) = 0$, determine the **free vibration motion** $y_1(t)$ when the initial conditions are $y_1(0) = 1\,\mathrm{m}$ and $\dot{y}_1(0) = 0$.

Note: Don't worry -- if you get the equations of motion incorrect (but is plausible), but your later work on c, d, e follows correctly after that, you will get full points for those later parts!

Hint: Feel free to use this video **https://www.youtube.com/watch?v=1TMoyVH_VLA** on your Carmen for inspiration (as well as your HW10 solutions, Q3 including the most recently posted update on Apr 30, here: **https://osu.instructure.com/courses/72915/files/21111088?module_item_id=4094257**).

(4) T_2 = spring force.

can we relate spring deflection to change in y_1.



$$\bar{V}_P = 0$$

$$\bar{V}_G = -R\omega\,\hat{j} = \dot{y}_1\,\hat{j}$$

$$\bar{V}_Q = -2R\omega\,\hat{j}$$

$$\bar{V}_S = -2R\omega\,\hat{j}$$

$$\Rightarrow \bar{V}_S = -2\dot{y}_1\,\hat{j}.$$

So S moves down by 2 units for every unit change in y_1.

$$T_2 = k\,(\text{spring deflection}) \uparrow$$

$$= +k(2\underbrace{\Delta y_1}_{\text{change in } y_1}).$$

if you have

$$(m)\,\ddot{y}_1 + (k)\,y_1 = C_1 + F_0 \cos \omega t.$$

General solution

$$y_1(t) = C_2 + A\cos(\omega_n t) + B\sin(\omega_n t) + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t.$$

ω_n - natural $= \sqrt{k/m}$

ω - given forcing frequency

where $C_2 = C_1/k$.