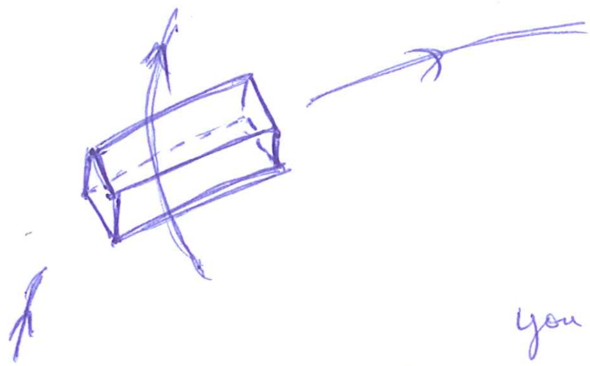
SPATIAL DESCRIPTIONS AND TRANSFORMATIONS

≡ 3D.

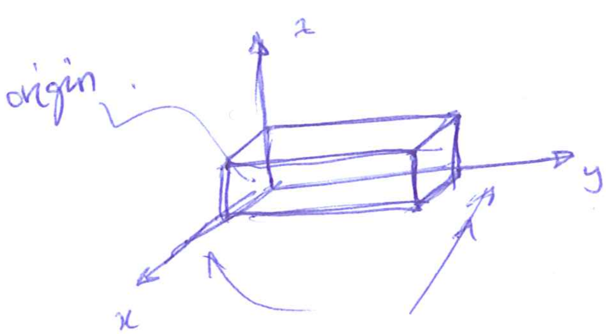
(CHAPTER 2)

Robotics is (mostly) about moving objects, so we need a language for describing the position and orientation of objects.



SAY you have a blackboard eraser flying through the air. How would you describe its position and orientation if you had to?

One way is to attach 3 "axes" to the object, and track the "axes" and the origin.

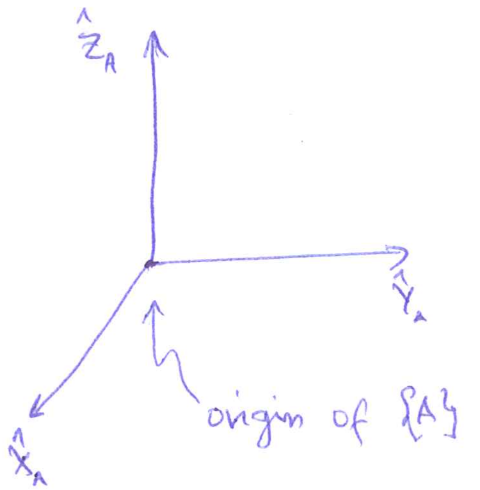


coffee stirrers posing as x, y, z axes

So it is good to think about axes, "reference frames", rotations, etc.

A <u>reference frame</u> is a coordinate system for describing the positions and orientation of (1) objects (2) other reference frames.

Eg.

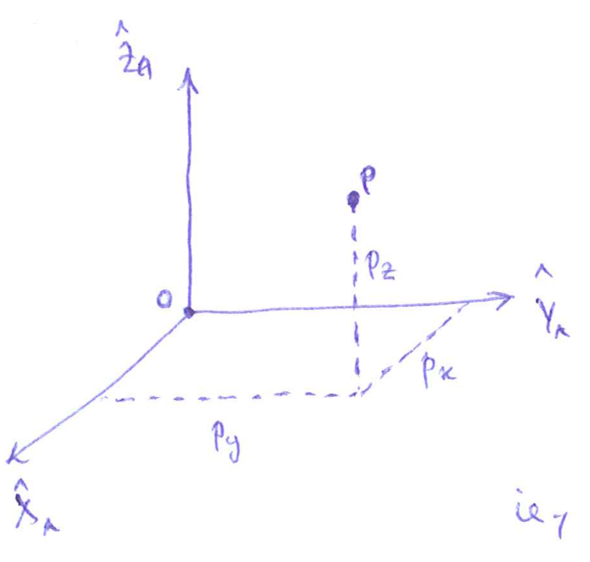


Reference frame = set of axes + origin.

<u>Notation</u>: $\{A\}$ = reference frame A.

$\hat{X}_A, \hat{Y}_A, \hat{Z}_A$ are <u>unit</u> vectors (length = 1) and mutually perpendicular.

<u>Position of some point P using $\{A\}$</u>



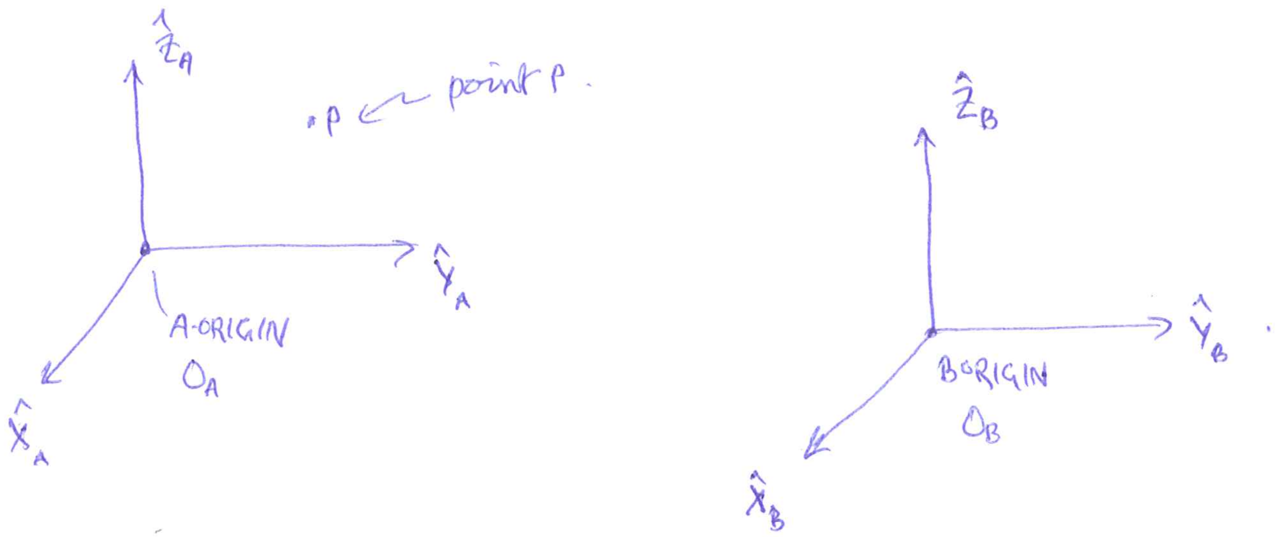
$$\text{vector } \vec{OP} = p_x \hat{X}_A + p_y \hat{Y}_A + p_z \hat{Z}_A .$$

Notation: ${}^A p$ = (vector) position of point P written in terms of the reference frame $\{A\}$.

ie, $${}^A p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$ is the <u>representation</u> of P in $\{A\}$

"Pure translations" (moving with rotation) of reference frames

Say $\{B\}$ is obtained by moving $\{A\}$ without rotation

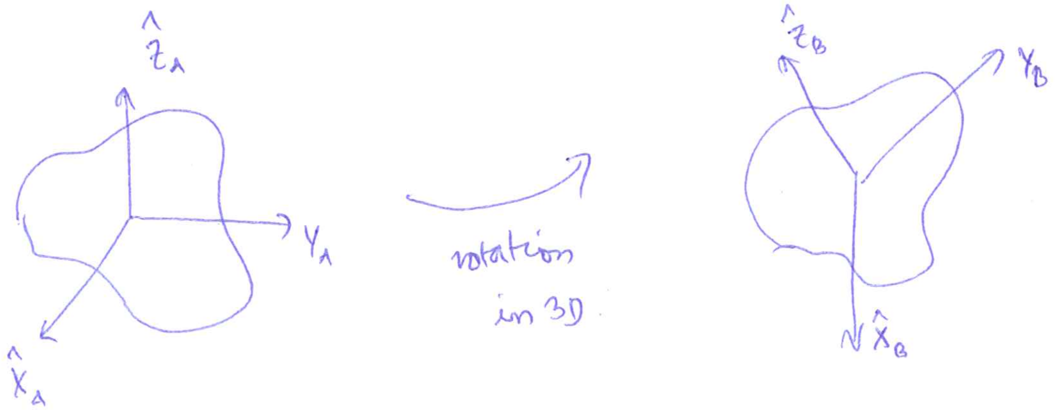


Given a representation of P in $\{A\}$ (namely ${}^A p$), how to obtain its representation in $\{B\}$?

$$\vec{O_A P} = \vec{O_A O_B} + \vec{O_B P}$$

$$\boxed{{}^A p = {}^B p + \underbrace{{}^A p_{B\text{-ORIGIN}}}_{\text{POSITION OF B-ORIGIN w.r.t. } \{A\}}}$$

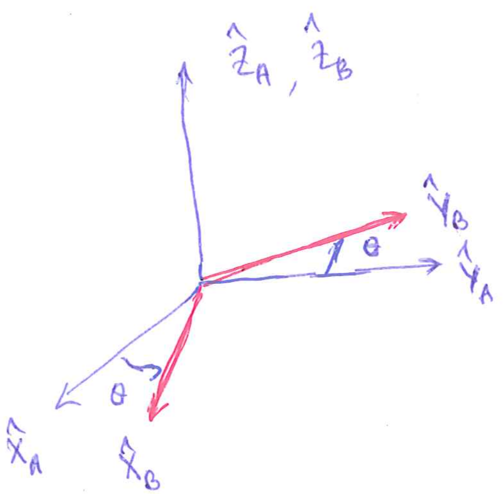
Rotations of frames (and how to represent them)



By tracking the position and orientation of a body-fixed set of axes, we can completely describe the motion of a RIGID body.

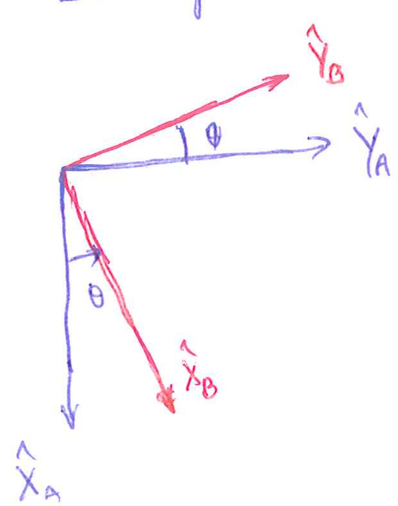
Simple rotation

(say about the $\hat{z}$ axis).



{B} obtained by rotating {A} by angle θ, about $\hat{Z}_A$ axis.

Looking down through $\hat{Z}_A$ ($\equiv \hat{Z}_B$).



What is the mathematical relation between the axes of $\{A\}$ and $\{B\}$?

$$\hat{X}_B = (\quad)\hat{X}_A + (\quad)\hat{Y}_A + (\quad)\hat{Z}_A$$

component of vector $\hat{X}_B$ along $\hat{X}_A$ axis

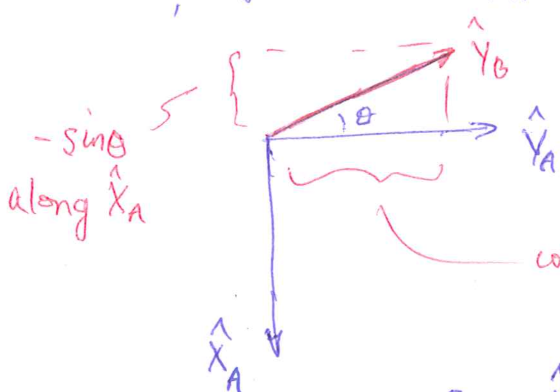
$= 1 \times \cos\theta$

Because length of $\hat{X}_B = 1$.

$1 \times \sin\theta$

0.

So, $\hat{X}_B = \cos\theta\,\hat{X}_A + \sin\theta\,\hat{Y}_A + 0\,\hat{Z}_A$.



So $\hat{Y}_B = -\sin\theta\,\hat{X}_A + \cos\theta\,\hat{Y}_A + 0\,\hat{Z}_A$.

$\hat{Z}_B = \hat{Z}_A$.

Equivalently, it can be shown that:

$\hat{X}_A = \cos\theta\,\hat{X}_B - \sin\theta\,\hat{Y}_B + 0\,\hat{Z}_B$.

$\hat{Y}_A = \sin\theta\,\hat{X}_B + \cos\theta\,\hat{Y}_B + 0\,\hat{Z}_B$

$\hat{Z}_A = \hat{Z}_B$.

This relation can be written in MATRIX notation.

$$\begin{bmatrix} \hat{X}_A \\ \hat{Y}_A \\ \hat{Z}_A \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{} \begin{bmatrix} \hat{X}_B \\ \hat{Y}_B \\ \hat{Z}_B \end{bmatrix}$$

Note: please review matrix multiplication to understand this.

This 3x3 matrix is an example of a "rotation matrix"

"Rotation matrices" are special kinds of matrices that encode the relationship between frames that are related to each other by rotations. Rotation matrices have special properties we'll briefly discuss later.

The book's notation for the above rotation matrix is ${}^{A}_{B}R$.

By our vocabulary/convention,

${}^{A}_{B}R$ is the "representation" of the frame $\{B\}$ in frame $\{A\}$.

and as above ${}^{A}_{B}R$ satisfies
$$\begin{bmatrix} \hat{X}_A \\ \hat{Y}_A \\ \hat{Z}_A \end{bmatrix} = {}^{A}_{B}R \begin{bmatrix} \hat{X}_B \\ \hat{Y}_B \\ \hat{Z}_B \end{bmatrix}$$

Similarly, $${}^{B}_{A}R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We note that $${}^{B}_{A}R = \left({}^{A}_{B}R\right)^{T}$$ at least for this simple rotation.

matrix transpose (ie, "reflecting" all the elements across the diagonal). Please review what this means.

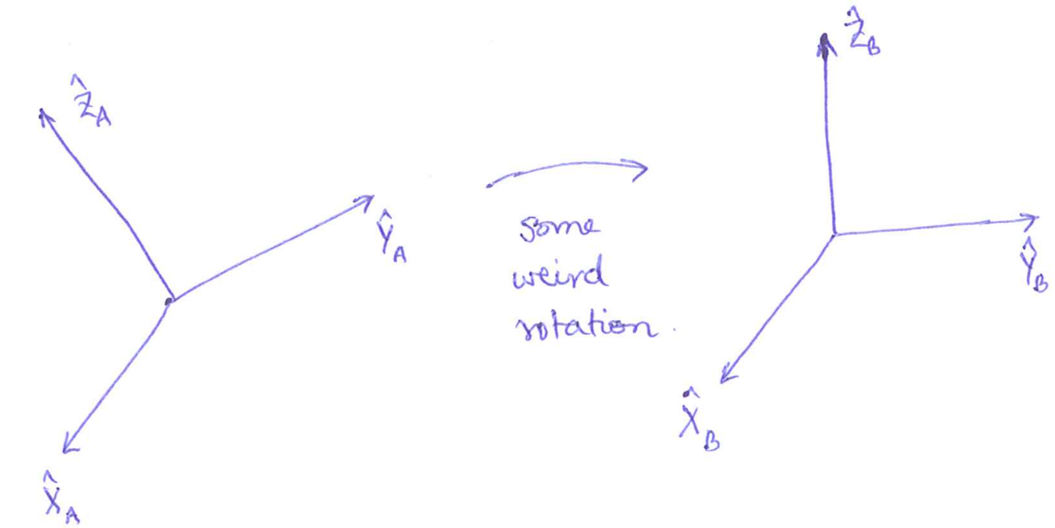
Rotation about $\hat{X}_A$ axis by angle θ (anti-clockwise = positive)

$${}^{A}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about $\hat{Y}_A$ by angle θ

$${}^{A}_{B}R = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

GENERAL ROTATION MATRICES



Say we have some general rotation (i.e., not necessarily along one of the three coordinate axes). What is the rotation matrix connecting $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$ and $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$. ?

Say we have some vector $\underline{v}$..

Then, $\underline{v}$ can be written in $\{A\}$ or $\{B\}$. as follows.

$$\underline{v} = (\,?\,)\hat{X}_A + (\,?\,)\hat{Y}_A + (\,?\,)\hat{Z}_A .$$

$$\underline{v} = \underbrace{(\,?\,)}_{\substack{\text{component of} \\ \underline{v}\text{ along} \\ \hat{X}_B \text{ (scalar} \\ \text{number}}}\hat{X}_B + \underbrace{(\,?\,)}_{\substack{\text{component} \\ \text{along } \hat{Y}_B}}\hat{Y}_B + \underbrace{(\,?\,)}_{\substack{\text{component} \\ \text{along } \hat{Z}_B}}\hat{Z}_B .$$

$$\underline{v} = (\underline{v}\cdot\hat{X}_B)\,\hat{X}_B + (\underline{v}\cdot\hat{Y}_B)\,\hat{Y}_B + (\underline{v}\cdot\hat{Z}_B)\,\hat{Z}_B . \quad (1)$$

Similarly, $\underline{v} = (\underline{v}\cdot\hat{X}_A)\hat{X}_A + (\underline{v}\cdot\hat{Y}_A)\hat{Y}_A + (\underline{v}\cdot\hat{Z}_A)\hat{Z}_A . \quad (2)$

Equations 1 and 2 are true for $\underline{\text{any}}$ vector $\underline{v}$.

In particular, we can use for $\underline{v}$ in Eqn 1.

$$\left.\begin{array}{rl} & \underline{v} = \hat{X}_A \\ \text{(or)} & \underline{v} = \hat{Y}_A \\ \text{(or)} & \underline{v} = \hat{Z}_A \end{array}\right\} \text{ to obtain expressions for } \hat{X}_A, \hat{Y}_A, \hat{Z}_A \text{ in } \{B\} \text{ respectively.}$$

So

$$\hat{X}_A = \overbrace{(\hat{X}_A \cdot \hat{X}_B)}^{\text{dot product = scalar number}} \hat{X}_B + (\hat{X}_A \cdot \hat{Y}_B)\, \hat{Y}_B + (\hat{X}_A \cdot \hat{Z}_B)\, \hat{Z}_B$$

$$\hat{Y}_A = (\hat{Y}_A \cdot \hat{X}_B)\, \hat{X}_B + (\hat{Y}_A \cdot \hat{Y}_B)\, \hat{Y}_B + (\hat{Y}_A \cdot \hat{Z}_B)\, \hat{Z}_B .$$

$$\hat{Z}_A = (\hat{Z}_A \cdot \hat{X}_B)\, \hat{X}_B + (\hat{Z}_A \cdot \hat{Y}_B)\, \hat{Y}_B + (\hat{Z}_A \cdot \hat{Z}_B)\, \hat{Z}_B .$$

$$\begin{bmatrix} \hat{X}_A \\ \hat{Y}_A \\ \hat{Z}_A \end{bmatrix} = \underbrace{\begin{bmatrix} (\hat{X}_A \cdot \hat{X}_B) & (\hat{X}_A \cdot \hat{Y}_B) & (\hat{X}_A \cdot \hat{Z}_B) \\ (\hat{Y}_A \cdot \hat{X}_B) & (\hat{Y}_A \cdot \hat{Y}_B) & (\hat{Y}_A \cdot \hat{Z}_B) \\ (\hat{Z}_A \cdot \hat{X}_B) & (\hat{Z}_A \cdot \hat{Y}_B) & (\hat{Z}_A \cdot \hat{Z}_B) \end{bmatrix}}_{{}^A_B R} \begin{bmatrix} \hat{X}_B \\ \hat{Y}_B \\ \hat{Z}_B \end{bmatrix}$$

${}^A_B R$ = this is the $\underline{\text{rotation matrix}}$ that takes $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$ to $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$

Rotation matrix ${}^A_B R$.

(1) – Helps us represent $\{B\}$ in terms of $\{A\}$.

(2) Gives us the transformation from $\{B\}$ to $\{A\}$.

If you defined {B} by both <u>translating</u> and <u>rotating</u> {A}, then {B} is fully described by

$$\left\{ \underbrace{{}^{A}_{B}R}_{\substack{\text{Rotation}\\ \text{matrix}}} \;,\; \underbrace{{}^{A}P_{B\text{-}ORIGIN}}_{\substack{\text{Displacement of origin of } \{B\}\\ \text{from that of } \{A\}.}} \right\}$$

Representation of a point P revisited (Now with ROTATIONS)

(1) Say {B} is obtained by a pure <u>translation</u> of {A}

Then, $^{A}p = {}^{B}p + {}^{A}P_{B\text{-}ORIGIN}$

(2) Say {B} is obtained by a <u>pure rotation</u> of {A}
(In other words, the origins of the 2 frames coincide).

Then $^{A}p = {}^{A}_{B}R\,{}^{B}p$.

(3) Say {B} is obtained by translation + rotation

Then $\boxed{{}^{A}p = {}^{A}_{B}R\,{}^{B}p + {}^{A}P_{B\text{-}ORIGIN}}$ ← IMPORTANT EQUATION

(INVERSE TRANSFORMATIONS)

NOTE 1: When ${}^{A}p = {}^{B}p + {}^{A}p_{BORG}$ (pure translation)

Then ${}^{B}p = {}^{A}p - {}^{A}p_{BORG} = {}^{A}p + {}^{B}p_{A\text{-}ORG}$.

NOTE 2: When ${}^{A}p = {}^{A}_{B}R\,{}^{B}p$, (pure rotation)

$${}^{B}p = \left({}^{A}_{B}R\right)^{-1}{}^{A}p = {}^{B}_{A}R\,{}^{A}p \quad \text{by convention}$$

matrix inverse of ${}^{A}_{B}R$.

$${}^{A}_{B}R^{-1} = {}^{B}_{A}R, \quad \text{by definition}$$

Review what a matrix inverse means and how it is computed

NOTE 3:

When ${}^{A}p = {}^{A}_{B}R\,{}^{B}p + {}^{A}p_{B\text{-}ORG}$ (translation + rotation)

$${}^{A}_{B}R\,{}^{B}p = {}^{A}p - {}^{A}p_{B\text{-}ORG}$$

$$\Rightarrow {}^{B}p = \left({}^{A}_{B}R\right)^{-1}\left({}^{A}p - {}^{A}p_{BORG}\right)$$

$$= {}^{B}_{A}R\left({}^{A}p - {}^{A}p_{BORG}\right).$$

Properties of all rotation matrices

(1) $\det\left({}^{A}_{B}R\right) = 1.$

(2) Most importantly,

$${}^{A}_{B}R \cdot \underbrace{{}^{A}_{B}R^{T}}_{\substack{\text{matrix}\\ \text{transpose}\\ \text{of } {}^{A}_{B}R.}} = I = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{the identity}\\ \text{matrix.}}}$$

That is, the rotation matrix multiplied with its transpose gives the identity matrix.

Properties (1) & (2) imply

$${}^{A}_{B}R \cdot \left({}^{A}_{B}R\right)^{T} = I$$

$$\underbrace{\left({}^{A}_{B}R\right)^{-1}\left({}^{A}_{B}R\right)}_{I} \cdot \left({}^{A}_{B}R\right)^{T} = \left({}^{A}_{B}R\right)^{-1} \cdot I$$

$$\Rightarrow \boxed{\left({}^{A}_{B}R\right)^{T} = \left({}^{A}_{B}R\right)^{-1}}$$

Inverse of a rotation matrix is its transpose!!

Note: This is a property only of rotation matrices and not general matrices, for which inverses are harder to compute.

Homogeneous Transformation :

We just saw that in general the representation of a point P in two coordinate frames {A} and {B}. are related by

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} \qquad - (3)$$

Can we rewrite in a "simpler" form that looks like

$${}^{A}P = {}^{A}_{B}T\,{}^{B}P \,? \qquad - (4)$$

Yes, by using the following trick. Consider the following equation:

$$\underbrace{\begin{bmatrix} \overset{3\times1\text{ matrix}}{{}^{A}P} \\ 1 \end{bmatrix}}_{4\times1\text{ matrix}} = \begin{bmatrix} \overset{3\times3\text{ matrix}}{{}^{A}_{B}R} & \overset{3\times1\text{ matrix}}{{}^{A}P_{BORG}} \\ \underset{1\times3\text{ matrix}}{0 \quad 0 \quad 0} & \underset{1\times1\text{ matrix}}{1} \end{bmatrix} \underbrace{\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}}_{4\times1\text{ matrix}} .$$

This is a matrix equation (5)

$$\boxed{\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}} \qquad - (6)$$

How is 5 the same as 3 ?

Multiplying out 5 we have:

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$ ~ same as 3

$$ 1 = [0\ 0\ 0]\,{}^{B}P + 1 = 1. $$ ~ silly equation 1=1.

That is, 5 is the same as 3 with an extra dummy equation that says 1=1.

In any case, this way of writing the translation + rotation is called "homogeneous transformation"

either written as

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}. $$

or as shorthand with an "implicit" 1 ; ${}^{A}P = {}^{A}_{B}T\,{}^{B}P$.

Of course, $ {}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] $ ← homogeneous transformation

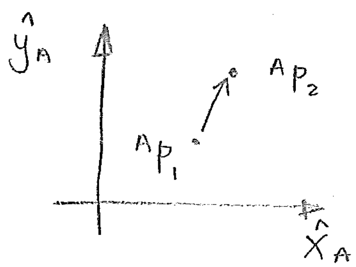
Of the form: (something) = (transformation) * (something else)

* Operators :

So far we have been using transforms to represent relations between frames and the representation of points in different frames.

Now we use the same ideas to move / rotate points, vectors, objects remaining in a single frame.

Pure translation :



Frame remains the same, move the point.

$$^A p_2 = {}^A p_1 + \underbrace{^A Q}_{\text{translation}}$$

$$^A p_2 = D_Q(q)\, {}^A p_1$$

displacement? by a vector Q.

where
$$D_Q(q) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

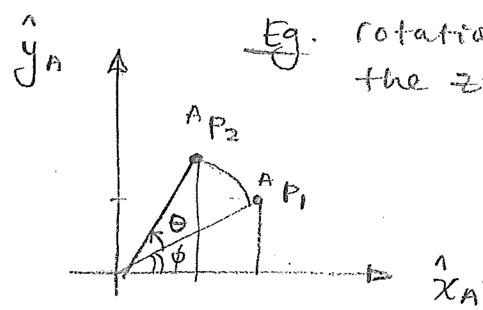
pure translation

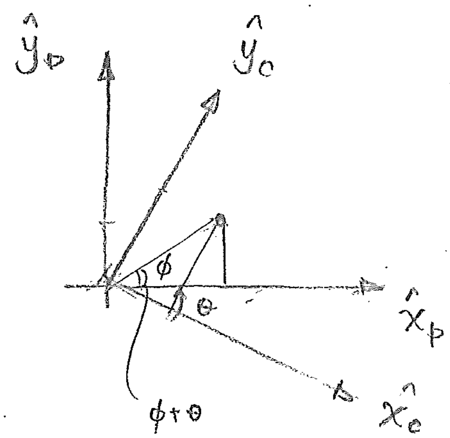
pure Rotation :

$${}^{A}P_2 = R \cdot {}^{A}P_1$$

①

We want ${}^{A}P_2$ given ${}^{A}P_1$





$${}^{C}P_1 = {}^{C}_{D}R \; {}^{D}P_1$$

②

① and ② are essentially identical and $R = {}^{C}_{D}R$.

$$R_z(\theta) = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

rotation about $\hat{z}$ by θ

pure rotation

From textbook:

" The rotation matrix that rotates vectors through some rotation R is the same as the rotation matrix that describes a frame rotated by R relative to the reference frame.

Transformation arithmetic:

$$
\begin{tikzcd}
\{U\} \arrow[r] \arrow[d] & \{A\} \arrow[r] & \{D\} \\
\{B\} \arrow[r] & \{C\} \arrow[r] & \{D\}
\end{tikzcd}
$$

$${}^{U}_{D}T = {}^{U}_{A}T \; {}^{A}_{D}T$$

$${}^{U}_{D}T = {}^{U}_{B}T \; {}^{B}_{C}T \; {}^{C}_{D}T$$

" Representation of D relative to frame C "

OR

" Transformation from D to C "

More on representing orientations/rotations:

So far we've used rotation matrices for representing rotations in 3D.

Superficially, a rotation matrix is $3 \times 3 = 9$ numbers. Rotations only require 3 numbers for complete specification.

The 9 numbers are all related in a manner that it takes only 3 numbers to specify all of them.

$$ {}^{A}_{B}R = \Big[\underbrace{{}^{B}\hat{X}_A}_{3\times1} \quad \underbrace{{}^{B}\hat{y}_A}_{3\times1} \quad \underbrace{{}^{B}\hat{z}_A}_{3\times1} \Big] $$

The 9 numbers / 3 vectors $(\hat{x}_A, \hat{y}_A, \hat{z}_A)$ are not independent

The vectors are unit magnitude and mutually orthogonal.

$$\left.\begin{array}{ll} |\hat{x}| = 1 & \hat{x}\cdot\hat{y} = 0 \\ |\hat{y}| = 1 & \hat{y}\cdot\hat{z} = 0 \\ |\hat{z}| = 1 & \hat{z}\cdot\hat{x} = 0 \end{array}\right\} \text{6 equations in 9 numbers.}$$

9 numbers
6 equations

---

3 independent numbers to represent a general rotation.

We can use 3 angles to represent a general 3D rotation. There are lots of ways to pick the angles you specify to describe a rotation.

24 different "fixed and Euler angle" conventions.

6

x-y-z fixed angles:

$\{A\}$ - fixed reference frame

obtain $\{B\}$ by:

(1) Rotation about (fixed) $\hat{X}_A$ by γ
(2) " " " $\hat{y}_A$ by β
(3) " " " $\hat{z}_A$ by α

12 different fixed angle conventions:

$$\left.\begin{array}{ll} xyz & xyx \\ yzx & xzx \\ zxy & yxy \\ yxz & yzy \\ xzy & zxz \\ zyx & zyz \end{array}\right\}$$

Any of these is a legal way to represent a 3D rotation.

Illegal: $\underbrace{xx}y$
they are not independent rotations!

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x-y-z Euler angles:

$\{B\}$ obtained from $\{A\}$ by:

(1) Rotation about $\hat{X}_B = \hat{X}_A$
(2) " " new $\hat{y}_B$
(3) " " new $\hat{z}_B$

Again, 12 legal permutations (same as before).

Angle sets to represent 3D rotations:

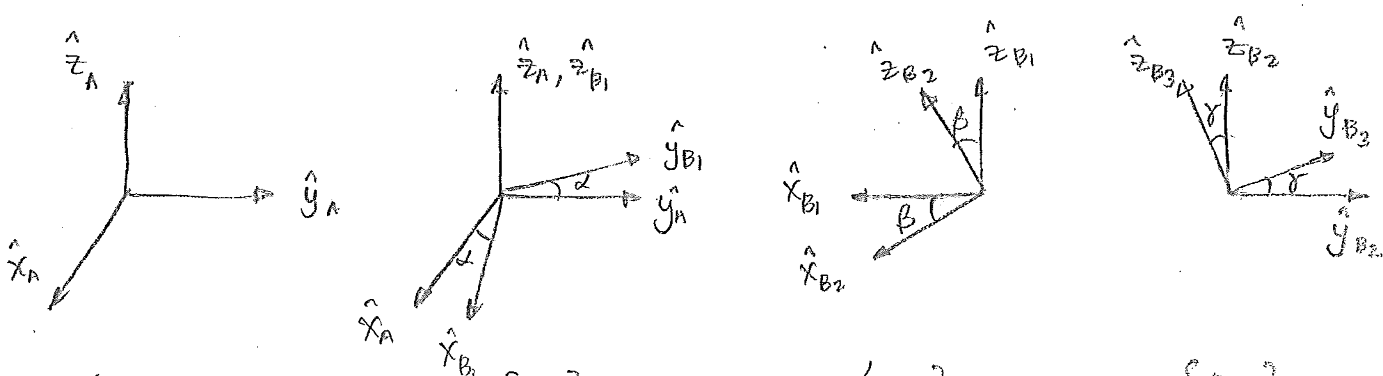
12 different Euler-angle sets.
12 different fixed angle sets.

Recall that general rotations require at most 3 numbers to specify.

z-y-x Euler angle convention:

$\{A\}$ - fixed reference frame (initially $\{B\} = \{A\}$)
The final frame $\{B\}$ is obtained by the following sequence of 3 rotations:

(1) about the $\hat{z}_B$ $(= \hat{z}_A)$ axis by α.
(2) about the new $\hat{y}_B$ axis by β
(3) about the new $\hat{x}_B$ axis by γ



$\{A\}$ $\{B_1\}$ $\{B_2\}$ $\{B_3\}$

$$\{A\} \rightarrow \{B_1\} \rightarrow \{B_2\} \rightarrow \underbrace{\{B_3\}}_{}$$

FINAL $\{B\} = \{B_3\}$

So the net rotation is:

$$ {}^{A}_{B}R = {}^{A}_{B_3}R = {}^{A}_{B_1}R \cdot {}^{B_1}_{B_2}R \; {}^{B_2}_{B_3}R $$

$$ = R_z(\alpha) \cdot R_y(\beta) \cdot R_x(\gamma) $$

where: $R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$ R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} $$

$$ R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & ?? \\ 0 & ?? & \cos\gamma \end{bmatrix} $$

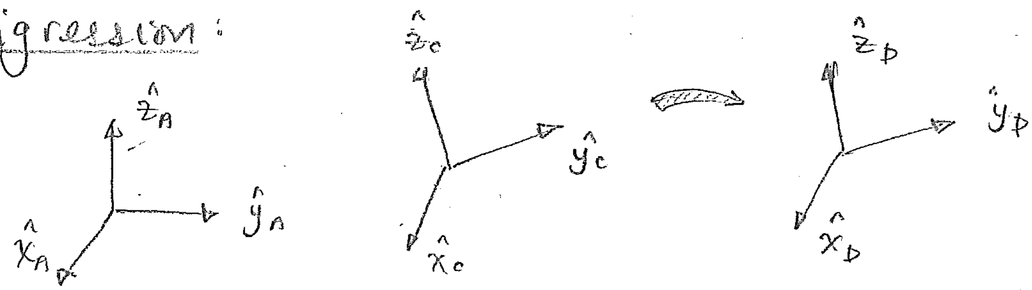
z-y-x fixed angle set:

$\{A\}$ - fixed reference frame

$\{B\} = \{B_3\}$ is obtained by the following rotations:

(1) Rotate about (fixed) $\hat{z}_A$ by angle α.
(2) " " " $\hat{y}_A$ " " β.
(3) " " " $\hat{x}_A$ " " γ.

Digression:



We obtain $\{D\}$ by rotating $\{C\}$ about some axis in $\{A\}$.

Apply the reverse rotation to $\{A\}$ to obtain $\{A_1\}$.

Then: ${}^{A_1}_{C}R = {}^{A}_{D}R$ i.e. the relation between A_1 and C is the same as the relation between A and D.

1st rotation:

So ${}^{A}_{D}R = {}^{A_1}_{C}R = {}^{A_1}_{A}R \; {}^{A}_{C}R \Rightarrow {}^{A}_{B_1}R = R_z(\alpha) \cancel{{}^{A}_{D}R}$

confusing!

If we have rotation about $\hat{z}$ by α

$${}^{A}_{A_1}R = R_z(\alpha)^T$$

$${}^{A_1}_{A}R = R_z(\alpha)$$

confusing!

$${}^{A}_{B_1}R = R_z(\alpha)\,\cancel{{}^{A}_{A}R} \quad \longleftarrow \quad 1^{st} \text{ rotation}$$

$${}^{A}_{B_3}R = R_x(\gamma)\,{}^{A}_{B_2}R \quad \longleftarrow \quad 1^{st}, 2^{nd}, 3^{rd} \text{ rotations.}$$

$$= R_x(\gamma)\left(R_y(\beta)\,{}^{A}_{B_1}R\right) \qquad \text{where} \quad {}^{A}_{B_2}R = R_y(\beta)\,{}^{A}_{B_1}R$$

$$= R_x(\gamma)\,R_y(\beta)\cdot R_z(\alpha)$$

1^{st}, 2^{nd} rotations.

NOTE: We not be using fixed angles!

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so far, we have discussed:

(1) Rotation matrices
(2) Euler angles
(3) Fixed angles.

Another representation:

Euler's theorem on rotations:

An orientation (3D) of a body can be obtained from any other orientation by a single rotation about some $\hat{k}$ by angle θ.

This theorem implies a representation of general rotations:

Equivalent angle-axis representation.

counting: $\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$ and $\theta \Rightarrow$ 4 numbers.

But the 4 numbers are not independent because $|\hat{k}| = 1$ (unit vector), i.e. $k_x^2 + k_y^2 + k_z^2 = 1$.

$$\Rightarrow k_z = \sqrt{1 - k_x^2 - k_y^2}$$

So specifying just three numbers is sufficient.

(k_x, k_y, θ) OR (k_y, k_z, θ)

can go from $\hat{k}, \theta$ to:

(1) Rotation matrices
(2) Euler angles
(3) etc.

} see book for various formulas for going from one representation to another.

Chapter 2 end (briefly again)

Manoj Srinivasan

Ways of representing 3D rotations (orientations)

- Rotation matrices (9 numbers)
- Fixed angles or Euler angles (3 numbers)
    - Based on the idea that any orientation can be obtained by rotating about 3 axes (any two consecutive axes not being identical)
    - 12 possible conventions for fixed or Euler angles, depending on which axes we rotate about.

Euler angles

Example: Z-Y-Z Euler angles

Original frame is $\{A\}$.
We get intermediate frame $\{B_1\}$ by rotating $\{A\}$ about $\hat{Z}_A$ by α.
We get intermediate frame $\{B_2\}$ by rotating $\{B_1\}$ about $\hat{Y}_{B1}$ by β.
We get final frame $\{B\}$ by rotating $\{B_2\}$ about $\hat{Z}_{B2}$ by γ.

Given the 3 angles, can get the rotation matrix as following:

$$\begin{aligned} {}^A_BR &= {}^A_{B1}R \cdot {}^{B1}_{B2}R \cdot {}^{B2}_{B}R \\ &= R_Z(\alpha) \cdot R_Y(\beta) \cdot R_Z(\gamma) \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \cdot \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Euler angles

Given the rotation matrix, can we get the corresponding (Z-Y-Z) Euler angles?

Say we know all the elements of the rotation matrix:

$$ {}^A_BR = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

Then:

$$\begin{aligned} \beta &= \operatorname{atan2}\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right) \\ \alpha &= \operatorname{atan2}\left(r_{23}/s\beta, r_{13}/s\beta\right), \\ \gamma &= \operatorname{atan2}\left(r_{32}/s\beta, -r_{13}/s\beta\right) \end{aligned}$$

where atan2() is essentially $\tan^{-1}()$,
but uses two arguments, namely the numerator and the denominator
to resolve which quadrant the angle should be in.

Yet another way to represent 3D orientations

Axis-Angle representation or "Equivalent angle-axis representation"

Based on the fact: any 3D orientation can be obtained from any other 3D orientation by a <u>single rotation</u> about an appropriately chosen axis

How many numbers is this?

one (1) for the angle
3 for the axis = 4 ...

really, 2 for the axis if unit vector. So 1+2 = 3 numbers

See book for specific formulas to get a rotation matrix from an axis and angle & vice versa

Miscellaneous useful things - Part 1.

+ part of lecture 7

Inverse of the homogeneous transform

Given
$$ {}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] $$

$$ {}^{B}_{A}T = \left[{}^{A}_{B}T\right]^{-1} = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] $$

See Eq. 2.45 in book

<u>Note</u>: ${}^{B}_{A}T \neq \left[{}^{A}_{B}T\right]^{T}$.

Deriving the rotation matrix for X-Y-Z Euler angles

The frame $\{B\}$ is obtained from $\{A\}$ by the following 3 rotations

$\{A\} \longrightarrow \{B_1\}$ by rotating about the axis $\hat{X}_A$ by angle α.

$\{B_1\} \longrightarrow \{B_2\}$ by rotating about the axis $\hat{Y}_{B_1}$ by angle β.

$\{B_2\} \longrightarrow \{B\}$ by rotating about the axis $\hat{Z}_{B_2}$ by angle γ.

$$ \{A\} \longrightarrow \{B_1\} \longrightarrow \{B_2\} \longrightarrow \{B\} $$

So ${}^{A}_{B}R = {}^{A}_{B_1}R\;{}^{B_1}_{B_2}R\;{}^{B_2}_{B}R$ and ${}^{A}_{B_1}R = R_X(\alpha)$, ${}^{B_1}_{B_2}R = R_Y(\beta)$,

and ${}^{B_2}_{B}R = R_Z(\gamma)$.

So $\quad {}^{A}_{B}R = R_x(\alpha)\, R_y(\beta)\, R_z(\gamma)$

Deriving the rotation matrix for X-Y-Z fixed angles

The frame $\{B\}$ is obtained from frame $\{A\}$ by (again) a sequence of 3 rotations, but now all about the fixed axes in $\{A\}$.

$\{A\} \longrightarrow \{B_1\}$. Rotation about $\hat{X}_A$, by angle α. (i)

$\{B_1\} \longrightarrow \{B_2\}$. Rotation about $\hat{Y}_A$, by angle β. (ii)

$\{B_2\} \rightarrow \{B\}$. Rotation about $\hat{Z}_A$, by angle γ. (iii)

Key fact : When we rotate a frame $\{C\}$ to obtain frame $\{D\}$, about an axis fixed in a third frame $\{A\}$, we have

$$ {}^{A}_{D}R = R \cdot {}^{A}_{C}R . \qquad -(iv)$$

$\uparrow$ rotation matrix in frame $\{A\}$.

Using (iv) for (i), (ii) and (iii), we have :

$$ {}^{A}_{B}R = R_z(\gamma)\, {}^{A}_{B_2}R \qquad -(v)$$

$$ {}^{A}_{B_2}R = R_y(\beta)\, {}^{A}_{B_1}R \qquad -(vi)$$

and $\; {}^{A}_{B_1}R = R_x(\alpha)$. (vii) $\leftarrow$ noting that ${}^{A}_{A}R = I$

Using (vi) and (vii) in (v), we have

$$
\begin{aligned}
{}^{A}_{B}R &= R_z(\gamma)\, {}^{A}_{B_2}R \\
&= R_z(\gamma) \left[R_y(\beta)\, {}^{A}_{B_1}R \right] \\
&= R_z(\gamma)\, R_y(\beta)\, R_x(\alpha)
\end{aligned}
$$

Note that this rotation matrix is very similar to the rotation matrix one obtains for Euler angles, except the 3 matrices are multiplied in reverse order. $R_x R_y R_z$

DEGREES OF FREEDOM

For some mechanical systems, the number of degrees of freedom can be a subtle concept.

But in this course, we will use the following definition:

The number of degrees of freedom of an object is the minimum number of independent parameters required to completely specify the position and orientation of all parts of the object. Colloquially, it is equal to the number of "directions" in which the parts of the body can move and rotate in.

Let us consider some examples.

Planar

1 DOF

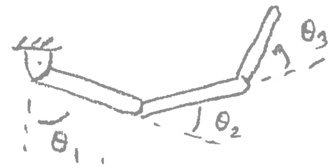
Specifying θ_1 is sufficient to describe the position & orientation of the body.

Planar 2-link

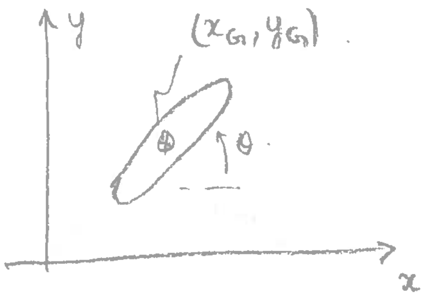
2 DOF

θ_1 and θ_2.

Planar 3-link

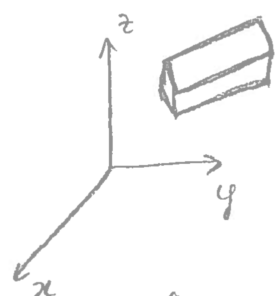


Single unconstrained rigid body in a plane.



3 DOF
(x_G, y_G, θ).

Single unconstrained rigid body in 3D

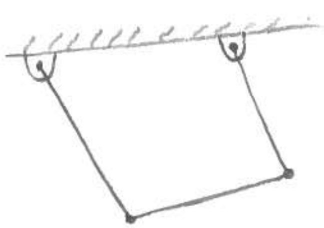


6 DOF

= 3 translational and 3 rotational DOF.

$$(x_G, y_G, z_G, \underbrace{\alpha, \beta, \gamma}_{\text{Some Euler angles}})$$

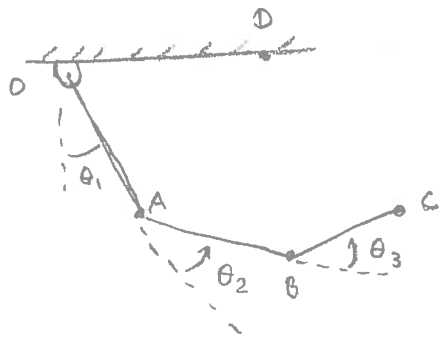
"Four bar" linkage



1 DOF

how to see that the 4 bar linkage has only one DOF?

· Start with a 3-link manipulator



When the point C is NOT attached to point D, the mechanism has 3 DOF (θ_1, θ_2 and θ_3).

When we attach point C on the link BC to point D on the ceiling, we introduce 2 constraints: namely

$x_C = x_D$.
and $y_C = y_D$.

These constraints make the $\theta_1, \theta_2, \theta_3$ related somehow. (dependent).

The 2 constraints reduce the number of DOF from 3 to 1.

$= 3 - 2 = 1$

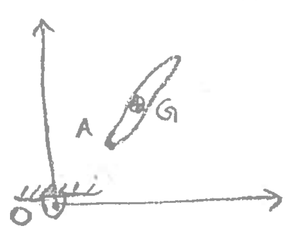
Revisit the single-link planar pendulum

We know that DOF = 1.

But let us get to this another way.

Start with an unconstrained rigid body in the plane.

It has 3 DOF (x_G, y_G, θ).



Now we attach the point A on the link to the point O. This introduces 2 constraints, so we are left with

$3 - 2 = 1$ DOF

[The 2 constraints make it so that x_G, y_G, θ are no longer independent, and x_G, y_G can be obtained given θ.]

- In this course, we will mainly consider objects (robot manipulators) that have <u>1 DOF per joint</u>.
  But in the <u>HW</u> we might consider <u>other</u> objects, more commonplace than robots, which do not necessarily have this property.