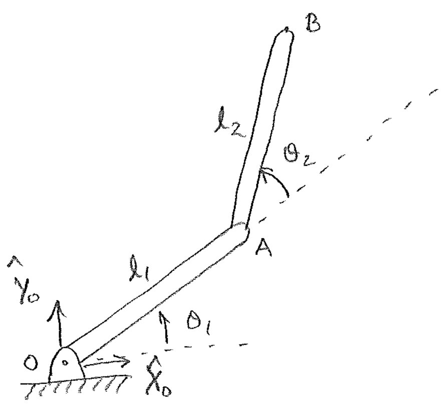
Lecture ME752 Differential kinematics

What's left in the course?

- Differential Kinematics - velocities / accerations, Singularities, Statics, etc.
- Dynamics - how forces and torques affect motion.
- Control - what forces and torques should be applied when to produce given motion or desired behavior.

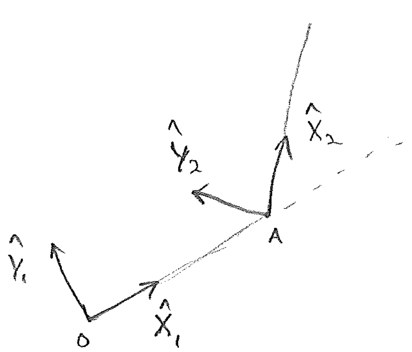
— Over the next few lectures, we will cover these topics in 2D so we are able to get to control. And then, if there is time, we can do some 3D differential kinematics & dynamics, as in the book.

Differential Kinematics



$${}^{0}P_A = \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \end{bmatrix}$$

This is the position of point A in frame 0.



$${}^{0}P_B = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \end{bmatrix}.$$

Both these expressions follow from basic geometry from the first couple of lectures.

"FORWARD" DIFFERENTIAL KINEMATICS

Now we wish to compute the velocities of points A and B with respect to the frame {0} and represented in frame {0}

Differentiating ${}^{0}P_A$ and ${}^{0}P_B$ wrt time and using the chain rule, we have

$$ {}^{0}v_A = \frac{d}{dt}\left({}^{0}P_A\right) = \begin{bmatrix} -\, l_1 \sin\theta_1 \cdot \dot{\theta}_1 \\ +\, l_2 \cos\theta_1 \cdot \dot{\theta}_2 \end{bmatrix} $$

$$ {}^{0}v_B = \frac{d}{dt}\left({}^{0}P_B\right) = \begin{bmatrix} -l_1 \sin\theta_1 \,\dot{\theta}_1 - l_2 \sin(\theta_1+\theta_2)\,(\dot{\theta}_1+\dot{\theta}_2) \\ l_1 \cos\theta_1 \,\dot{\theta}_1 + l_2 \cos(\theta_1+\theta_2)\,(\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix} $$

Collecting terms

$$ {}^{0}v_B = \begin{bmatrix} -\left(l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)\right)\dot{\theta}_1 - l_2 \sin(\theta_1+\theta_2)\,\dot{\theta}_2 \\ \left(l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)\right)\dot{\theta}_1 + l_2 \cos(\theta_1+\theta_2)\,\dot{\theta}_2 \end{bmatrix} $$

Rewriting as a matrix product

$$ = \begin{bmatrix} -\left(l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)\right) & -\,l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \times \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ = J(\Theta)\,\dot{\Theta} $$

where $\Theta = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$ vector of joint angles. Θ - UPPERCASE THETA

and

$$ J(\Theta) = \begin{bmatrix} -\left[l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)\right] & -\,l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} $$

$J(\Theta)$ is called the "Jacobian" or the "Jacobian matrix".

Given the joint angle rates $\dot{\Theta}$, the Jacobian matrix $J(\Theta)$ gives the velocity of some point on the manipulator.

In robotics,
Even though Jacobians are usually used in the context of the end-effector / end-point, each point might be associated with its own Jacobian.

For instance, the Jacobian that relates ${}^0v_A$ to $\dot{\Theta}$ is given by

$$ {}^0v_A = \begin{bmatrix} -l_1 \sin\theta_1 \, \dot{\theta}_1 \\ +l_1 \cos\theta_1 \, \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 & 0 \\ +l_1 \cos\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

So for A, $J_A(\Theta) = \begin{bmatrix} -l_1 \sin\theta_1 & 0 \\ +l_1 \cos\theta_1 & 0 \end{bmatrix}$.

MATH ASIDE : What's a Jacobian?

The Jacobian is not a concept specific to robotics. It is not always tied to velocities and angle rates.

A Jacobian is just a generalization of the notion of a derivative to vector-valued functions of vectors.

Say $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ is a function of $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

$$y_1 = f_1(x_1, x_2, \ldots x_n)$$
$$y_2 = f_2(x_1, x_2, \ldots x_n)$$
$$\ldots$$
$$y_m = f_m(x_1, x_2 \ldots x_n)$$

Say x_i are function of t (which could be time, or any other variable)
Then, to find $\dot{Y}$ given $\dot{X}$, we can use the chain rule of differentiation:

$$\frac{dy_1}{dt} = \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}\dot{x}_2 + \ldots + \frac{\partial f_1}{\partial x_n}\dot{x}_n$$

$\vdots$ and so on.

$$\frac{dy_m}{dt} = \frac{\partial f_m}{\partial x_1}\dot{x}_1 + \frac{\partial f_m}{\partial x_2}\dot{x}_2 + \ldots + \frac{\partial f_m}{\partial x_m}\dot{x}_n.$$

$$\dot{Y} = J(X)\,\dot{X} \quad , \quad \frac{dY}{dt} = J(X)\frac{dX}{dt}.$$

$$J(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

matrix of all partial derivatives

- Jacobian of Y with respect to X.

If we use $Y = {}^0P_B$ and $X = \Theta$, we get the Jacobian for end point velocity ${}^0v_B$.

→ See MATLAB demo in class. and related programs.

Can use the command <u>jacobian</u> to get the Jacobian.

* first, we re-did the Jacobian / expression for ${}^0v_B$ for the two-link manipulator.

* next, we computed the Jacobian / end point velocity for the non-planar RRR manipulator of Lecture 10.

Thus, at least to get end-point velocity, 3D is essentially the same as 2D (conceptually).

– Usually for 3D manipulators, we usually not only want the <u>end point velocity</u>, but also the <u>end-effector angular velocity</u>.

<u>INVERSE DIFFERENTIAL KINEMATICS</u>
(<u>INVERSE INSTANTANEOUS KINEMATICS</u>)

FORWARD : Given joint angle rates, what is the end point velocity?

$$\boxed{v_B = J(\Theta)\,\dot{\Theta}}$$

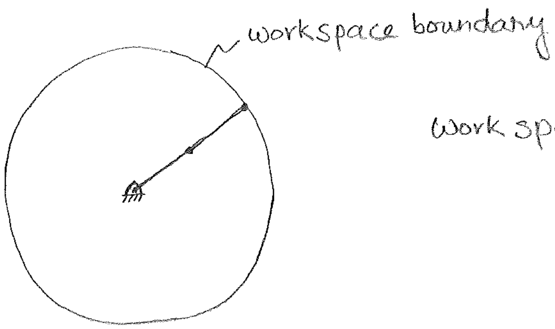
INVERSE : Given end point velocity, find the joint angle rates required.

$$\boxed{\dot{\Theta} = J(\Theta)^{-1}\,v_B}$$

If the Jacobian matrix $J(\Theta)$ is <u>invertible</u>, we can find $\dot{\Theta}$ from v_B.

When is $J(\Theta)$ likely to be non-invertible? (while remaining within the reachable workspace)

Eg: For the two-link manipulator, the Jacobian becomes non-invertible at the <u>boundary</u> of the reachable workspace.



workspace boundary is defined by $\theta_2 = 0$ (outer)
$\theta_2 = \pi$ (inner boundary)

$$J(\Theta) = \begin{bmatrix} -(l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

At $\Theta = \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix}$, we have $J\left(\begin{bmatrix} \theta_1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\ l_1 \sin\theta_1 + l_2 \sin\theta_1 & -l_2 \sin\theta_1 \\ l_1 \cos\theta_1 + l_2 \cos\theta_1 & l_2 \cos\theta_1 \end{bmatrix}$

$$J\left(\begin{bmatrix} \theta_1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -(l_1+l_2)\sin\theta_1 & -l_2 \sin\theta_1 \\ (l_1+l_2)\cos\theta_1 & l_2 \cos\theta_1 \end{bmatrix}$$

Notice that the two columns of J are simple multiples of each other.

This means (1) determinant (J) = 0
(2) J is not invertible
(3) rank (J) < 2

all equivalent.

$\Rightarrow$ SINGULAR

whatever the number of DOFs.

If you try to compute the inverse of J in MATLAB, you would get one of

$$\begin{bmatrix} \text{Inf} & \text{Inf} \\ \text{Inf} & \text{Inf} \end{bmatrix} \text{ (or) perhaps } \begin{bmatrix} \text{NaN} & \text{NaN} \\ & \dots \end{bmatrix}$$

(Not a number)

What does all this mean, physically?

Let us compute the end point velocity at the singularity.

$$^0v_B = J(\theta)\,\dot{\Theta} = \begin{bmatrix} -(l_1+l_2)\sin\theta_1\,\dot{\theta}_1 - l_2\sin\theta_1\,\dot{\theta}_2 \\ (l_1+l_2)\cos\theta_1\,\dot{\theta}_1 + l_2\cos\theta_1\,\dot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta_1\left(+(l_1+l_2)\dot{\theta}_1 + l_2\dot{\theta}_2\right) \\ \cos\theta_1\left((l_1+l_2)\dot{\theta}_1 + l_2\dot{\theta}_2\right) \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 \cdot q \\ \cos\theta_1 \cdot q \end{bmatrix}$$

where $q = (l_1+l_2)\dot{\theta}_1 + l_2\dot{\theta}_2$

$$^0v_B = q\begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}$$

$${}^0v_B = q \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}.$$

What does this mean?

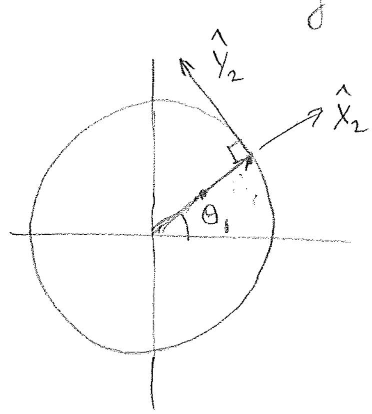
(1) Whatever the values of $\dot{\theta}_1$ and $\dot{\theta}_2$ (some finite numbers), the end point velocity is always along $\begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}$ direction at the boundary (outer) of the workspace.

That is, NOT all velocity directions are possible.

This is a defining feature of all singularities

[In Linear algebra Speak]: The range space of the Jacobian has dimension < 2 $\neq 2$.

(2) It can be shown that the direction $\begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}$ is along the tangent to the circle, namely the workspace boundary.

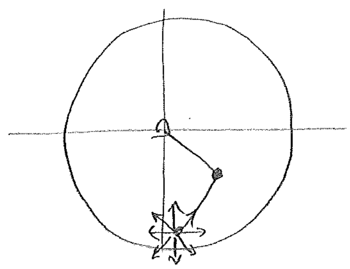


$$\boxed{\hat{Y}_2 = -\sin\theta_1 \hat{X}_0 + \cos\theta_1 \hat{Y}_0}$$

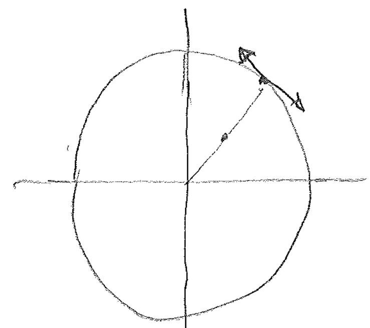
and $\hat{X}_2 = \cos\theta_1 \hat{X}_0 + \sin\theta_1 \hat{Y}_0$.

In other words, when on the workspace boundary, you cannot move in or out with finite speed.

Only along the boundary at finite speed.



Velocity in any direction is possible if inside the reachable workspace (not on boundary)



On the boundary of reachable workspace, only tangential velocities are possible.

At singularity, $\det(J) = 0$.

Near singularity, $\det(J) \approx 0$.

$\Rightarrow$ near a singularity $\mathrm{inv}(J) \approx \begin{bmatrix} \text{some} \\ \text{LARGE} \\ \text{numbers} \end{bmatrix}$ but not quite infinity.

$\Rightarrow$ for some output velocities v_B, we might need large $\dot{\Theta}$ to achieve them.

because $\dot{\Theta} = \mathrm{inv}(J)\, v_B$.

Large $\dot{\Theta}$ will be needed if v_B has a non-zero component along the radial direction.

Eg. $l_1 = 1$, $l_2 = 1$.

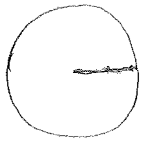
<u>at singularity</u> $\theta_1 = 0$, $\theta_2 = 0$.

$$J = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad \det(J) = 0 - 0 = 0.$$

$$\text{inv}(J) = \begin{bmatrix} \text{Inf} & \text{Inf} \\ \text{Inf} & \text{Inf} \end{bmatrix}.$$

<u>near singularity</u> $\theta_1 = 0$, $\theta_2 = 0.01$, say.

(close to boundary)

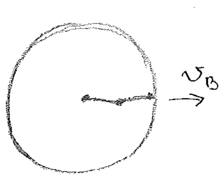


$$J \approx \begin{bmatrix} -0.01.. & -0.01 \\ 2.00 & 1.00 \end{bmatrix}, \quad \det(J) = 0.01$$ (small but not zero)

$$\Rightarrow \text{inv}(J) = \begin{bmatrix} 99.997 & 1.000 \\ -199.998 & -1.000 \end{bmatrix}.$$ ← Some LARGE numbers.

$\Rightarrow$ For <u>some</u> output velocities v_B, $\dot{\Theta}$ can be large.

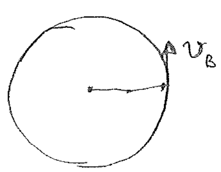
First, consider ${}^0v_B$ close to the "radial" direction



$$v_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\dot{\Theta} = \text{inv}(J) * {}^0v_B = \begin{bmatrix} 99.9967 \\ -199.998 \end{bmatrix}$$ ← need LARGE joint rates.

Next, consider ${}^0v_B$ close to the "tangential" direction.



$${}^0v_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\dot{\Theta} = \text{inv}(J) * {}^0v_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ ← do not need LARGE joint angle rates

For more complex mechanisms with more DOFs

Singularities can be one of two types:

(1) Workspace-boundary singularities
(2) Workspace-interior singularities

usually caused by lining up of 2 or more joint axes.

the number of output degrees of freedom (or velocity directions) decrease at a singularity.

So singularities are mostly avoided in robot design.

Robots are often designed to work in parts of the workspace, bounded away from a singularity.

Examples are bent-legged walking robots (like old Asimo and BigDog robots). But humans use straight legs. Why?

[At singularities, robots lose local controllability]

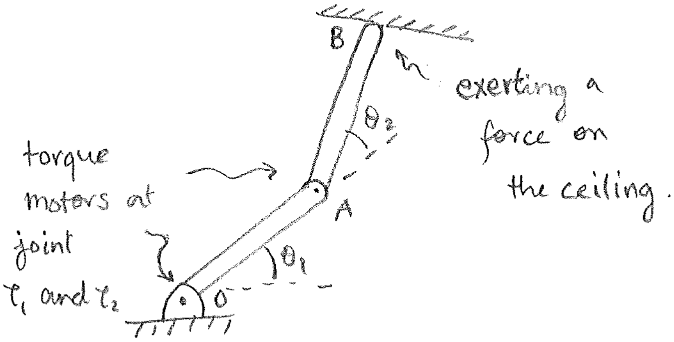
But we'll soon see how singularities can sometimes be taken advantage of.

```
% Using MATLAB for jacobians, velocities and accelerations
theta = [theta1; theta2; theta3]
thetadot = [theta1dot; theta2dot; theta3dot]
thetadotdot = [theta1dotdot; theta2dotdot; theta3dotdot]
% position
P = some function of the theta1, theta2, etc. derived from forward dynamics
% velocity
v = jacobian(P,theta)*thetadot; % just chain rule ...
% acceleration
a = jacobian(v,thetadot)*thetadotdot + jacobian(v,theta)*thetadot
```

The Jacobians for more complicated 3d robots can similarly be obtained using symbolic MATLAB, if we have an expression for the position of the point in 3d. We just need to differentiate this expression.

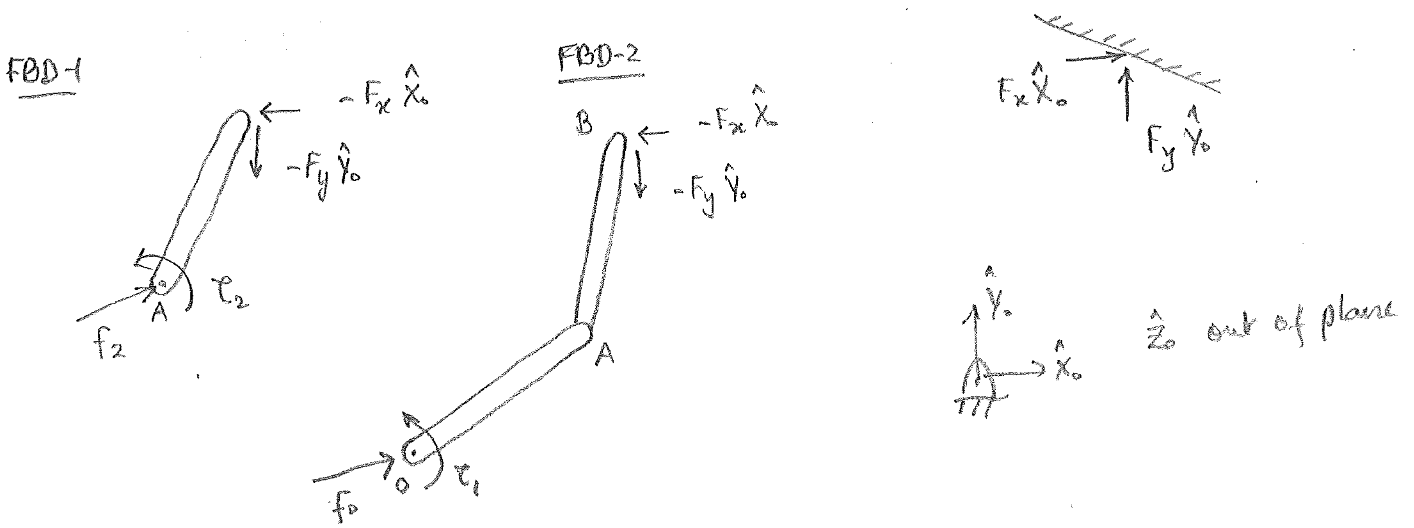
STATICS IN 2D (and 3D)

Let us start with a simple example, the planar 2D manipulator. But remarkably, the result we'll obtain is general and applicable in 3D.



What is the relationship between the external force exerted at the end point B, and the joint torques? if the manipulator is at REST.

We can obtain this relation by using force balance or moment balance on appropriate free body diagrams.



Take moments about A for FBD1:

$$\tau_2 \hat{Z}_0 - F_y \left(L_2 \cos(\theta_1+\theta_2)\right)\hat{Z}_0 + F_x \left(L_2 \sin(\theta_1+\theta_2)\right)\hat{Z}_0 = 0$$

Take moments about O for FBD2:

$$\tau_1 \hat{Z}_0 - F_y \left[L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2)\right]\hat{Z}_0 + F_x \left[L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2)\right]\hat{Z}_0 = 0.$$

We can write these two equations in matrix form:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -[L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2)] & L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) \\ -L_2 \sin(\theta_1+\theta_2) & L_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

Note from previous lecture that the Jacobian for the endpoint B

is $J = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2) & -L_2 \sin(\theta_1+\theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) & L_2 \cos(\theta_1+\theta_2) \end{bmatrix}$.

which is just the transpose of the above matrix..

That is, we have shown:

$$\boxed{\tau = J^T F}$$

This is the relation between the joint torques and end point force.

This relation is true more generally, for a n-link manipulator in 3D. So to get the relation between the torques and the end point forces, we only need to compute the jacobian, which we know how to do in 3D.

Thus, you know enough now to do 3D statics in the absence

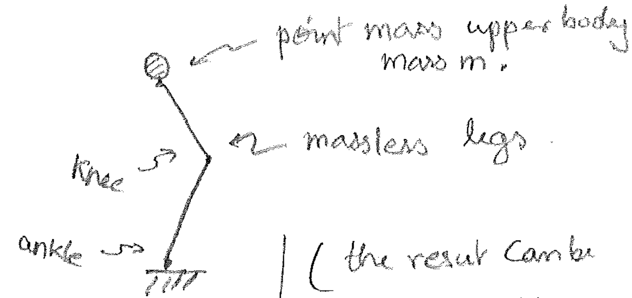
Note that the above equation is a robot design tool. Typically what is specified are theexterna

Example. Why is it much more tiring to stand with bent legs?

Simple Model of a person

not so tiring   tiring.

why?

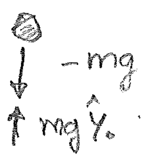


$L_1 = L_2 = L$

(the result can be shown without these simplifications as well).

Say we have torque motors at the knee (τ_2) and the ankle (τ_1)

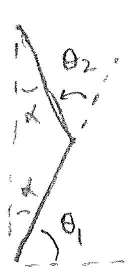
This simple model of the human body is identical to our 2-link manipulator.



$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 0 \\ mg \end{bmatrix}.$$

The "external force" exactly cancels body weight, so is $mg\hat{Y}_0$ - the force the leg exerts on the upper body.

Also, we might assume that the hip is directly above the ankle and this implies a relation between θ_1 and θ_2.



By geometry, $\theta_2 = 2\alpha$.
and $\alpha = 90° - \theta_1$

So $\theta_2 = 2(90° - \theta_1) = 180° - 2\theta_1$.

$\Rightarrow \theta_1 + \theta_2 = \theta_1 + 180° - 2\theta_1 = 180° - \theta_1$

We have

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2) & L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) \\ -L_2 \sin(\theta_1+\theta_2) & L_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} 0 \\ mg \end{bmatrix}.$$

force exerted by leg on upper body.

$$= L \begin{bmatrix} -\sin\theta_1 - \sin\theta_1 & \cancel{L}\cos\theta_1 - \cancel{L}\cos\theta_1 \\ -\sin\theta_1 & -\cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ mg \end{bmatrix}.$$

using $\theta_1+\theta_2 = 180^\circ - \theta_1$

$$= L \begin{bmatrix} -2\sin\theta_1 & 0 \\ -\sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} 0 \\ mg \end{bmatrix}.$$

$$= \begin{bmatrix} 0 \\ -mgL\cos\theta_1 \end{bmatrix}.$$

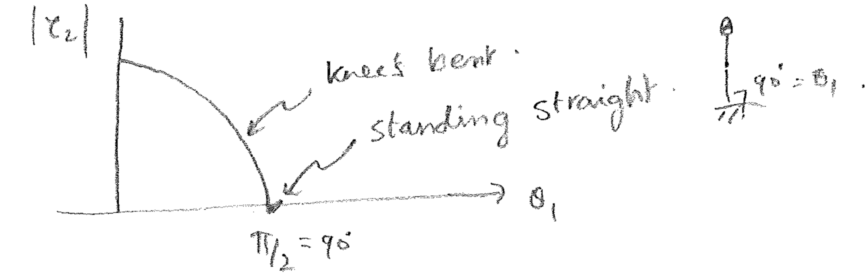
That is, $\boxed{\begin{aligned} \tau_1 &= 0 \\ \tau_2 &= -mgL\cos\theta_1 \end{aligned}}$ ← ankle torque $= 0 = \tau_1$ as a consequence of hip directly above heel.

$|\tau_2| = |\text{Knee torque}| = mgL\cos\theta_1$. ← needs to be provided by "knee muscle" group.

When knee is straight, $\theta_1 = 90^\circ$

$|\tau_2| = mgL\cos 90^\circ = 0$

When knee is bent $|\tau_2| \neq 0$.



So if you want your knees to not tire, use $\theta_1 = 90°$ and $\theta_2 = 0$.
i.e., stand with the knees straight.

$\theta_2 = 0$, knees straight is a <u>singularity</u>. If your goal is to exert forces in the "radial" direction, it is best to be as close to this singularity as possible. Theoretically, at the singularity, the mechanism can produce arbitrarily high vertical forces, with essentially zero joint torques.

This situation is the reverse of the relation between endpoint velocity and joint angle rates where we wanted to avoid singularities.

Thus, whether we <u>like</u> or <u>dislike</u> singularities depends on the goal, i.e., whether we want to produce <u>forces</u> or <u>motion</u> in certain directions at the end point!!

(This is true in 3D as well).