Homework 4, Nonlinear Dynamics, Spring 2016

Manoj Srinivasan

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Discrete Dynamical Systems

Q1. [10 pts]. The following one-D discrete dynamical system is called the **tent map**.

$$x_{n+1} = f(x_n) = \begin{cases} \mu x_n, & \text{for } x_n < \frac{1}{2} \\ \mu(1 - x_n), & \text{for } x_n \geq \frac{1}{2} \end{cases}$$

For all the questions below, just consider the x range $[0, 1]$, whether for plotting or for initial conditions.
a) Plot $f(x)$ vs x for different values of μ, say $\mu = 0.5, 1, 1.5,$ and 2. All on the same plot, using hold on and axis equal.
b) Determine all fixed points and their stability for the range $\mu = 0$ to 4.
c) To find 2-periodic motions, we can consider the dynamical system:

$$x_{n+2} = f(f(x_n))$$

Write this in the form $y_{n+1} = g(y_n)$ where $y_{n+1} = x_{2n+2}$ and $g(x) = f(f(x))$. Fixed points of this new dynamical system in y, which are not fixed points of the original system $f(x)$, will be 2-periodic motions for the original dynamical system $f(x)$. Find all such fixed points for $\mu \in (0, 2)$ and determine their stability, thus determining the stable and unstable 2-periodic motions and their range for the original dynamical system.
d) Draw an orbit diagram for the tent map by plotting the list of x values at steady state versus μ. This is kind of a bifurcation diagram. This is analogous to the orbit diagram for the logistic map.

Q2. [5 pts]. Consider the 2-D nonlinear discrete dynamical system: $x_{i+1} = x_i - (y_i - 1)^2$ and $y_{i+1} = 5 - x_i - y_i^2$, where x and y are real numbers. Determine the fixed points, determine the Jacobian at the fixed point, find its eigenvalues and comment on the local stability of the fixed point.

Q3. How good is the Euler's method? [7 pts]. All numerical methods for solving ODEs take a continuous time system and convert it to a discrete time system. For example, given a one-D differential equation of the form $\dot{x} = g(x)$, the Euler's method essentially involves using the following iteration $x_{i+1} = x_i + g(x_i)\Delta t = f(x_i)$, where $x_i = x(i\Delta t)$ and Δt is the integration step-size.

Consider the differential equation $\dot{x} = kx$. Derive a closed-form expression for the discrete dynamical system $x_{i+1} = f(x_i)$ obtained when the Euler's method is used on this differential equation. For what values of Δt and k is this discrete dynamical system stable? Is the stability behavior of the discrete dynamical system for a given k always same as that of the original ODE? If not, when is this the case? Is this a problem?

Analysis such as this underlie the development of more sophisticated "stable" numerical methods for solving differential equations.

Q4. Finding the square root of a number. [8 pts]. If you did not have a calculator or a computer, how would you find the square root of a real number a?

a) Here is a simple iterative method, which perhaps some of you learnt in high school:

$$x_{i+1} = 0.5 \left(x_i + \frac{a}{x_i} \right) \tag{1}$$

Explain why this algorithm is a reasonable way to find square roots. When does it work?
b) What about the following different algorithm?

$$x_{i+1} = x_i + x_i^2 - a. \tag{2}$$

Does it work? Does it always work?
c) You all learnt Newton-Raphson method for finding roots of equations of the form $f(x) = 0$. Apply Newton-Raphson to the equation $x^2 - a = 0$. You can then treat the resulting algorithm as a discrete dynamical system. Compare to the methods from parts-a and b. When do we have convergence to the root?