ME8230
Nonlinear Dynamics

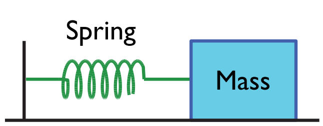
Lecture 1, part 1

Introduction, some basic math background, and some random examples

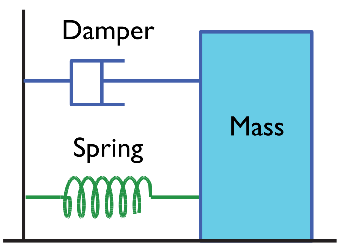
Prof. Manoj Srinivasan
Mechanical and Aerospace Engineering
srinivasan.88@osu.edu

Spring mass damper system
LINEAR

(unforced)



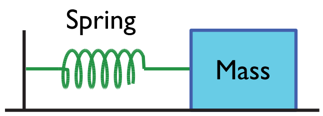
$m\ddot{x} + kx = 0$ (undamped)



$m\ddot{x} + c\dot{x} + kx = 0$ (damped)

(You can add "external forcing terms" to all the "unforced" examples in this lecture)

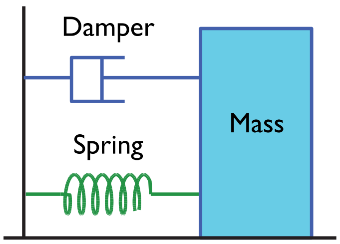
Spring mass damper system NON-LINEAR



undamped $m\ddot{x} + f(x) = 0$

f(x) = Nonlinear spring force

g(xdot) = Nonlinear damping

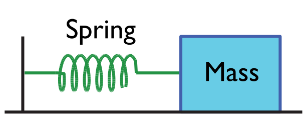


damped $m\ddot{x} + g(\dot{x}) + f(x) = 0$

more general nonlinear damped mass-spring system $m\ddot{x} + h(x, \dot{x}) = 0$

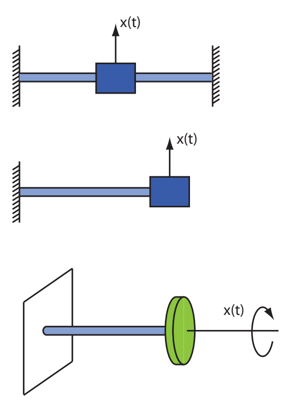
(unforced)

Nonlinear spring-mass examples



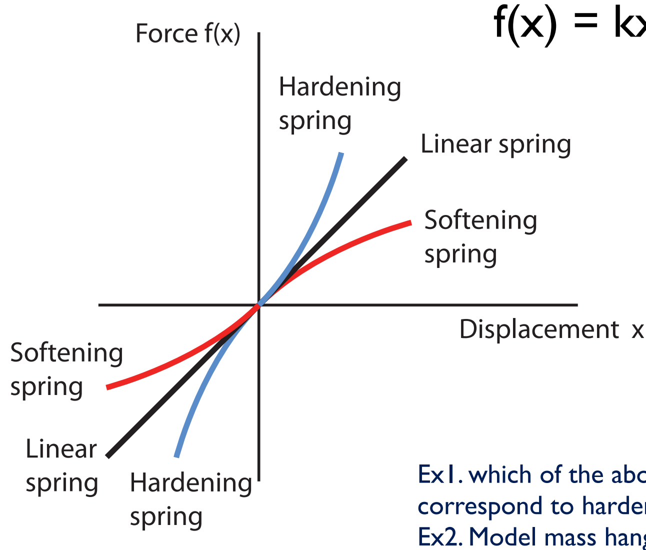
$m\ddot{x} + f(x) = 0$

can be a <u>nonlinear</u> model of any of these systems on the right <u>when the displacements are large enough</u> ... (but mostly dominated by the first mode)



Displacements small enough implies Linear spring approximation would be good enough (usually).

Common Nonlinear Stiffness behaviors



f(x) = kx+ax^3 , a>0, k>0

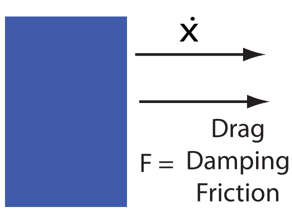
f(x) = kx

f(x) = kx-ax^3

Different nonlinearities might lead to different dynamical behaviors

Ex1. which of the above cubic f(x) correspond to hardening and softening springs?
Ex2. Model mass hanging from a taut horizontal string.

Nonlinear damping examples



Linear damping: $F = -c\dot{x}$

(negative sign indicates drag opposes velocity)

Fluid drag at large Re: $F = -c\dot{x}|\dot{x}| = -c(\dot{x})^2 \mathrm{sign}(\dot{x})$

Coulomb friction: "dry friction"
$$\text{if } \dot{x} = 0, \ -\mu_s N \leq F \leq \mu_s N$$
$$\text{if } \dot{x} \neq 0, \ F = -\mu_d N \ \mathrm{sign}(\dot{x})$$

Material damping at high enough strain rates is nonlinear

Many degrees of freedom systems

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$$

LINEAR

$$M\ddot{X} + C\dot{X} + KX = 0$$

Equations obtained
from FEM, modal analysis, etc.

K - stiffness matrix
M - mass matrix
C - damping matrix

E.g., trusses/beam/plate/other
elastic structures undergoing
small deformations

NON-LINEAR

$$M\ddot{X} + H(X, \dot{X}) = 0 \longleftarrow$$

Pretty general: most smooth nonlinear (discrete, unforced) mechanical systems have such equations

Forced finite dimensional mechanical systems

$$M(X)\,\ddot{X} = G(t, X, \dot{X})$$

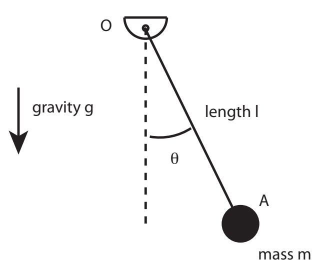
pretty
general
mechanical system

$$M(X)\ddot{X} = G_1(X, \dot{X}) + G_2(t)$$

special case
e.g., double pendulum
with joint torques

Other Examples

Simple pendulum



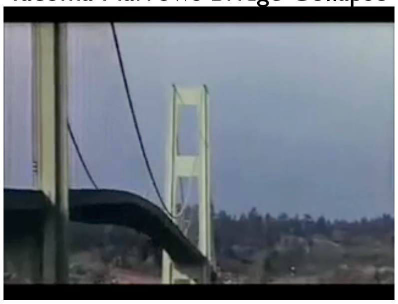
$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Do you know what the solutions
to this equation looks like?
Hint: Not sines and cosines in general.

Aeroelastic oscillations

Tacoma Narrows Bridge Collapse



NASA Tail Flutter Test

Nonlinear (negative) damping

"Hopf bifurcation"

"Limit cycles"

Youtube videos of
Tacoma Narrows Bridge Collapse
and NASA tail flutter test

Motions of disks and cylinders

Equations of motion for a cylinder rolling without slip

$$Q_{i1}\ddot{\psi} + Q_{i2}\ddot{\phi} + Q_{i3}\ddot{\theta} = S_i, \quad i = 1,2,3. \qquad (1)$$

where

$$Q_{11} = A \sin\phi - mHR\cos\phi + mH^2 \sin\phi,$$

$$Q_{12} = 0, \quad Q_{13} = -mHR,$$

$$Q_{21} = Q_{23} = 0, \quad Q_{22} = -mR^2 - mH^2 - A,$$

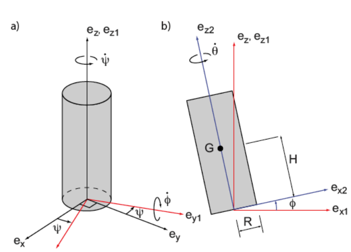
$$Q_{31} = C\cos\phi + mR^2\cos\phi - mRH\sin\phi,$$

$$Q_{32} = 0, \quad Q_{33} = C + mR^2,$$

$$S_1 = (C - 2A - 2mH^2)\dot{\psi}\dot{\phi}\cos\phi + C\dot{\phi}\dot{\theta} - 2mHR\dot{\psi}\dot{\phi}\sin\phi,$$

$$S_2 = (C - A + mR^2 - mH^2)\dot{\psi}^2 \sin 2\phi/2 + (C + mR^2)\dot{\theta}\dot{\psi}\sin\phi$$
$$+ mHR\dot{\psi}\dot{\theta}\cos\phi + mHR\dot{\psi}^2\cos 2\phi$$
$$+ mg(R\cos\phi - H\sin\phi),$$

$$S_3 = C\dot{\psi}\dot{\phi}\sin\phi + 2mR\dot{\psi}\dot{\phi}(R\sin\phi + H\cos\phi).$$



Paper by Srinivasan and Ruina
See demo, video and simulation.

Double pendulum

(Double pendulum video: Strogatz)

See also MATLAB example

(Hamiltonian) Chaos
Sensitive dependence on initial conditions
Lyapunov exponents

Double pendulum with torque actuators at each joint and PID control

“Robot arm”

See MATLAB example

Animations of controlled robot arm (double/triple pendulum)

Robots and humans

Hybrid (piecewise smooth) systems
Non-smooth systems
Limit cycles
Stability, Linearized
Collisions

Videos:
Passive dynamic robots
Spring-mass hopper

ODEs corresponding to
mechanical systems tend to be
(most naturally)
2nd order ODEs

Highest derivative order = 2

High (e.g., 2nd) order ODEs to first order ODEs

always possible
to convert

Example

Say you are given 2nd order ODE: $m\ddot{x} + kx = 0$

Introduce new variables
for all derivatives of x,
except the highest order (=2)

$$x_1 = x$$
$$x_2 = \dot{x}$$

Write an ODE for each
of the new variables

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1/m\end{aligned}$$

What is this course about?

Mostly, ordinary
differential equations
of the form

$$\frac{dy}{dt} = f(y,t)$$

$$\begin{aligned} t &= \text{independent variable} \\ y &= \text{dependent variable} \\ &\quad \text{(can be scalar or vector)} \end{aligned}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

What to do with PDEs?

Partial Differential Equations

Simplify, Discretize, Project

(e.g., finite elements, finite volume, finite differences, other Galerkin projections, lumped parameter approximations)

$\longrightarrow$

Ordinary Differential Equations

Some of the ideas in the course actually carry over to PDEs without reducing them to ODEs, but we will not discuss such.

Linear ODEs

$$\frac{dy}{dt} = f(y,t)$$

Linear time-varying inhomogeneous (non-homogeneous) "variable coefficients"	Linear time-varying homogeneous (variable coefficients)	Linear time-invariant homogeneous (constant coefficients)
$\frac{dy}{dt} = A(t)y + B(t)$	$\frac{dy}{dt} = A(t)y$	$\frac{dy}{dt} = Ay$
Inhomogeneous means B(t) is not zero	Homogeneous means B(t)=0	A is a constant and not time-varying

Linear superposition for linear homogeneous ODE

$$\frac{dy}{dt} = A(t)y$$

If $y_1(t)$ and $y_2(t)$ are solutions,
then $y_3 = y_1(t) + y_2(t)$ is also a solution.

$$\begin{aligned} \frac{dy_1(t)}{dt} &= A(t)y_1(t) \\ \frac{dy_2(t)}{dt} &= A(t)y_2(t) \end{aligned} \xrightarrow{\text{add}} \begin{aligned} \frac{d(y_1(t)+y_2(t))}{dt} &= A(t)y_1(t) + A(t)y_2(t) \\ &= A(t)\left(y_1(t)+y_2(t)\right) \end{aligned}$$

$$\text{Thus } \frac{y_3(t)}{dt} = A(t)y_3(t)$$

Linear superposition for linear inhomogeneous ODE

$$\frac{dy}{dt} = A(t)y(t) + B(t)$$

B(t) - Input (the "inhomogeneous term")
y(t) - Output

Say $y_1(t)$ is a solution when $B_1(t)$ is the input.
Say $y_2(t)$ is a solution when $B_2(t)$ is the input.

$y = \lambda_1 y_1 + \lambda_2 y_2$ is a solution when $B = \lambda_1 B_1 + \lambda_2 B_2$.

$$\begin{aligned} \dot{y}_1 &= A(t)y_1 + B_1(t) \\ \dot{y}_2 &= A(t)y_2 + B_2(t) \end{aligned} \xrightarrow{\text{add}} \dot{y}_1 + \dot{y}_2 = A(t)(y_1+y_2) + (B_1(t)+B_2(t))$$

Key properties of linear systems

- Superposition of solutions
- proportionality (input and output scale together).
- Various analytical and semi-analytical techniques available for periodic, non-periodic forcing. In the absence of forcing, "closed-form" solutions known. e.g., Laplace transforms, Fourier series, Green's functions, modal analysis, etc.

Nonlinear systems have none of these nice properties, in general. This course is about making sense of nonlinear ODEs by other means.