LINEARIZING ABOUT A LIMIT CYCLE

Consider the nonlinear differential equation

$$\dot{x} = f(x) \quad \text{with } \underset{\sim}{x} \in \mathbb{R}^n \quad f: \mathbb{R}^n \to \mathbb{R}^n . \quad - \ (1)$$

vector ODE.

Suppose that it has a limit cycle solution $x^*(t)$.

This means (1) $x^*(t) = x^*(t+T)$ where T is the period of the limit cycle. $x(t)$ is periodic.

(2) $\dot{x}^* = f(x^*)$ because that's what means to say $x^*(t)$ is a solution of the ODE

Let us try to <u>linearize</u> the system about the periodic motion $x^*(t)$.

$$\dot{x} = f(x)$$
$$= f(x^*) + J(t) \cdot (x - x^*(t)) + \text{HOT} \quad - (2)$$

higher order terms.

where $J(t)$ is the Jacobian of $f(x)$ with respect to x evaluated at $x^*(t)$. It is an $n \times n$ matrix. i.e. the $(i,j)^{th}$ element of J will be the partial derivative $\dfrac{\partial f_i}{\partial x_j}$.

Because $x^*(t)$ is a periodic function, $J(t)$ is a time-periodic matrix. That is, $\boxed{J(t) = J(t+T)}$

Now, let us define,

$$Y = X - X^* \quad \text{in } (2)$$

$$\dot{X} = \cancel{\dot{Y} + \dot{X}^*} = f(\cancel{X^*(t)}) + J(t) \cdot Y .$$

$$\Rightarrow \boxed{\dot{Y} = J(t) \cdot Y} \quad - (3)$$

Thus the linearization about a limit cycle is a linear time-periodic system. (Recall, we got linear time-invariant systems for linearization about fixed points). Linear time-periodic is a special case of linear time-varying.

For the linearization (3), $Y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a fixed point which corresponds to the limit cycle $x^*(t)$.

The stability of fixed point determines the stability of the limit cycle.

NOTE : Even If the eigenvalues of $J(t)$ always had negative real parts for all time t. one may or not have stability.

LINEARIZING ABOUT A PERIODIC SOLUTION OF A TIME-DEPENDENT ODE

Consider a time-varying ODE

$$\dot{x} = f(t,x) \quad \text{with } x \in \mathbb{R}^n, \; f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n. \qquad (1)$$

This equation has, as special cases:

- $\dot{x} = g(x)$. Autonomous / time-invariant ODE.
- $\dot{x} = g(x) + h(t)$. simple forced dynamics

In any case, say equation (1) has a periodic solution $x^*(t)$. That is $\dot{x}^* = f(t, x^*(t))$. — (2)

Define $y = x - x^*(t)$ — (3)

and Taylor-expand (1) about x^*.

$$\dot{x} = f(t, x^*) + J(t)\,(x - x^*) + HOT$$

$$= \dot{y} + \dot{x}^*(t)$$

$$\Rightarrow \boxed{\dot{y} = J(t)\, y(t)} \quad \text{linearization}$$

~~again J(t) is time~~ $\quad J_{ij}(t) = \dfrac{\partial f_i}{\partial x_j}$ matrix

It is not immediately clear whether $J(t)$ is time-periodic because $\frac{\partial f}{\partial x}$ has its own time dependence which has to "match" the period T if J is to be time-periodic.

(Counter-intuitive Example)

WARNING

$\dot{Y} = J(t)Y$ can have an unstable fixed point ~~at~~ $Y^* = 0$ even if $J(t)$ has only negative real parts for all time.

Example $J(t) = \begin{bmatrix} 1-4(\cos 2t)^2 & 2+2\sin 4t \\ -2+2\sin 4t & 1-4(\sin 2t)^2 \end{bmatrix}$

(1) You can show that $J(t)$ has the eigenvalue characteristic equation, $\lambda^2 + 2\lambda + 1 = 0$, independent of time t. So that $\lambda_1 = -1$, $\lambda_2 = -1$ for all time.

(2) We can also verify that

$$Y(t) = \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}$$ is a solution of the ODE

$\dot{Y} = J(t)Y$.

This is an unbounded solution ($\to \infty$) as $t \to \infty$

Thus, this is an example for ~~which~~ which $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is unstable even though $J(t)$ always has eigenvalues with only negative real parts.

— Example from Nemytskii and Vinograd / G.A. Leonov / Fritz Colonius's notes at IMA.

FLOQUET THEORY.

The subject of the solution and stability of linear time-periodic systems is called Floquet theory due to Gaston Floquet (French mathematician, 1883).

- Consider the ODE $\dot{Y} = J(t)\, Y$ — ①

  with $J(t) = J(t+T) \in \mathbb{R}^{n\times n}$.

  $Y \in \mathbb{R}^{n}$.

- Because of the time-periodicity of $J(t)$, if $Y(t)$ is a solution of ①, so is $Y(t+T)$. — ②

- Consider the $n\times n$ identity matrix

$$Z(0) = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & 0 & 1 & 0 \\ 0 & & & 0 & 0 & 1 \end{bmatrix} \quad — ③$$

Now we define $Z(t) \in \mathbb{R}^{n\times n}$, as the matrix whose j^{th} column is the solution $Y(t)$ to equation ① when $Y(0)$ the initial condition is the j^{th} column of matrix $Z(0)$ in ③

Another way of stating the same thing is that $\dot{Z}(t) = J(t) \cdot Z(t)$ with $Z(0) = \text{eye}(N)$.

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- The columns of $Z(t)$ will always be linearly independent $\forall t$

The columns of $Z(0)$ are linearly independent, by definition.

Thus, we can write the columns of $Z(T)$ as linear combinations of $Z(0)$.

$$Z(T) = Z(0) \cdot \underbrace{C}$$

$n \times n$ matrix that does the relevant linear combination

$$\Rightarrow \boxed{Z(T) = C} \qquad \text{because } Z(0) = \text{identity}.$$

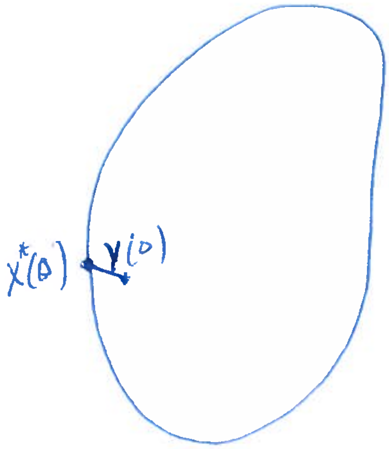
More generally,

$$\boxed{Z(t+T) = Z(t) \cdot C}$$

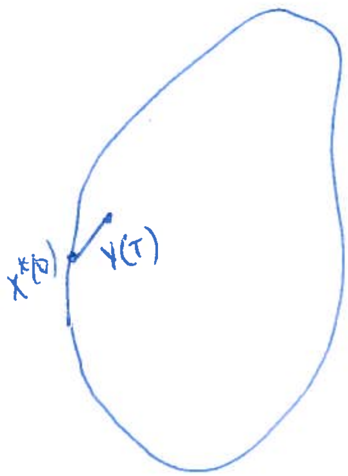
What is C?

$C = Z(T)$ is the matrix whose columns are obtained by integrating the ODE n-times, with n- initial conditions $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix}$ etc. for exactly time T.

Meaning of the monodromy matrix C



- Start at $x^*(0)$ on the limit cycle, integrate forward for time period T, we get back to $x^*(0)$.
- Start at $x^*(0) + Y(0)$ and integrate forward for time T, we will get back to a different point $x^*(0) + Y(T)$.



- the monodromy matrix C gives the transformation from $Y(0)$ to $Y(T)$.

$$Y(T) = C\,Y(0).$$

~~The monodromy matrix C does not depend on the initial condition X(0) we pick on the periodic motion~~

~~Because~~ $\cancel{z(t+T) = z(t)\cdot C = z(t)\cdot z(T)}$

Stability theorem

if all the eigenvalues of C have modulus (absolute value, real or complex) less than unity, then

$$y^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ is asymptotically stable.}$$

if even one of the eigenvalues of C has absolute value greater than 1, the origin is unstable.

Intuition:

$$z(t+T) = z(t) \cdot C.$$

$$z(t+mT) = z(t) \cdot \underbrace{C^m}.$$

if $C \in \mathbb{R}$, this would go to zero if $|c| < 1$.

for $m \to \infty$.

So that $z \to 0$.

Result / Proof

$C^m \to 0$ if all eigenvalues of C have absolute value less than 1.

Say C has n distinct eigenvectors such that

$$D = R^{-1} C R$$ is a diagonal matrix

with R's columns being the eigenvectors of C.

Then $$C^2 = R^{-1} D \cancel{R} \cdot \cancel{R^{-1}} D R = R^{-1} D^2 R$$

More generally, $C^m = R^{-1} D^n R$

$$D^m = \begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}$$

$\lambda_i^m \to 0$ if $|\lambda_i| < 1$ and $m \to \infty$.

$\to [0]$ zero matrix.

$\Rightarrow$ $C^m = R^{-1} D^m R \to 0$ as $m \to \infty$ if all λ_i of C have $|\lambda_i| < 1$.

The Eigenvalues of the matrix C
are called Floquet multipliers.

If C is nxn, there are n Floquet multipliers
(n-dimensional state space).

If we are doing the stability analysis using
a Poincare map, n-1 of the Floquet multipliers
are identical to the eigenvalues of the n-1. dimensional
Poincare map. IF the limit cycle comes from a time
independent ODE.

Recall. If you have a limit-cycle for a time-independent ODE
in n-dimensions, the Poincare section is an
n-1 dimensional surface and the Poincare map
takes a point on this n-1 dimensional surface
back to itself.

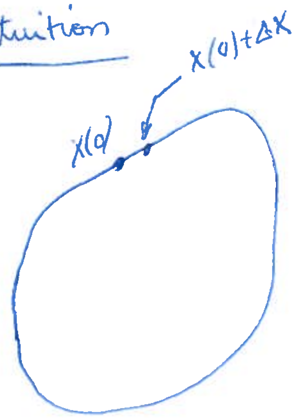
So there are n-1 eigenvalues
of the linearization to the Poincare map.

The n^{th} Floquet multiplier is equal to 1, if there
is no time dependence in the original ODE.

There is an

Unit eigenvalue for the monodromy matrix when $\dot{Y} = J(t)\,Y$ is derived by linearizing an autonomous ODE $\dot{X} = f(X)$ about a limit cycle

~~Intuition~~ Intuition



If you start from $X(0)$ on the limit cycle and integrate forward for time T.

If you start from $X(0) + \Delta X$, which is ALSO on the limit cycle.

then we expect that, if we integrate forward for time T, we'd get back to $X(0)+\Delta X$.

Given that the monodromy matrix C tells us how small perturbations transform after one cycle T.

we have $\Delta X = C\,\Delta X$

ie, ΔX is an eigenvector with eigenvalue 1 for the matrix C.

Note 1. ΔX is along the limit cycle trajectory at $X(0)$.

Theorem / Proof 2 | . The full time-dependent solution to $\dot{y} = J(t)\,y$ has a simple form (noted below at ⊛)

C is called the monodromy matrix.

This is a constant matrix.

~~Say~~ There exists a matrix B, possibly complex valued, such that

$e^{TB} = C$. where e^{TB} is the matrix exponential of TB.

T ← scalar, B ← $n \times n$ matrix,

B is a constant matrix.

We appreciate that

$$Z(t+T) = Z(t) \cdot C = Z(t) \cdot e^{TB} \quad - ①$$

We would like to "solve" for $Z(t)$ – in the sense that we would like $Z(t) =$ something that does not depend on Z.

'Ansatz' ("Guess") Let us look for solution $Z(t)$ of the form:

$$Z(t) = P(t) \cdot e^{tB} \quad - ②$$

Use ② in ①

$$Z(t+T) = P(t+T) \cdot e^{(t+T)B} = P(t) \cdot e^{TB} \cdot e^{tB}$$

$$\Rightarrow \quad P(t+T)\, \cancel{e^{(t+T)B}} = P(t)\, \cancel{e^{(t+T)B}}$$

Thus a solution of the form ② is valid if $P(t+T) = P(t)$. ie, $P(t)$ is a periodic matrix.

Thus

$Z(t)$ has a solution of the form

$$Z(t) = P(t) e^{tB}$$ where $P(t)$ is a periodic matrix with period T. ⍟

In other words, once we know the solution for one period, ~~everything~~ the solution is defined for all time.

Because $P(t)$ is continuous & bounded,

$$Z(t) \to 0 \text{ if } e^{tB} \to 0 \text{ as } t \to \infty.$$

It can be shown that

$$e^{tB} \to 0 \text{ as } t \to \infty$$

$$\text{if all } eig(C) < 1.$$

For instance, ~~consider~~
we have shown that

$$C^n \to 0 \text{ as } n \to \infty \text{ if all } |\lambda_i| < 1$$

we can write $C^n = (e^{TB})^n = e^{(nT \cdot B)}$

$$\lim_{n\to\infty} e^{nT \cdot B} = \lim_{t\to\infty} e^{tB} = 0.$$

Note 1. $P(0) = I$. because $Z(0) = I = P(0) \cdot e^0 = P(0) = I$.

Note 2: We have not described how to compute $P(t)$, only that such a function exists that satisfies above properties.

ideonotes

Existence of B such that $e^{TB} = C$

Say, as before,

$$D = R^{-1} C R, = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow C = R D R^{-1}$$

Also, a matrix exponential of a real matrix is always invertible.

Aside: **Matrix exponential background.**

$$e^A = ? = I + \frac{A}{1!} + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots$$

if A is diagonalizable $A = HGH^{-1}$ where G is diagonal.

$$\Rightarrow e^A = I + \underbrace{\frac{HGH^{-1}}{1!}}_{HGH^{-1}} + \frac{1}{2!} HG^2H^{-1} + \dots = H \underbrace{e^G}_{\text{diagonal matrix!}} H^{-1}$$

~~Then~~ ~~Back to~~ Now, clearly we can write the diagonal matrix $D = e^F$ where F is a diagonal matrix

if $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$, $F = \begin{bmatrix} \ln \lambda_1 & & \\ & \ln \lambda_2 & \\ & & \ddots \end{bmatrix}$. ← not unique necessarily if complex numbers are allowed.

Then, $C = R e^F R^{-1}$

$$= e^{RFR^{-1}} = e^{TB}$$

$$\boxed{B = \frac{1}{T} \left(RFR^{-1} \right)}$$

This B will satisfy $e^{TB} = C$

MORE ADVANCED

Further remarks :

EQUIVALENCE OF THE TIME-PERIODIC SYSTEM TO A TIME-INVARIANT SYSTEM

(1) Implicity, what we've shown so far transforms the time-periodic system $\dot{Y} = J(t)Y$ into a time-invariant system. Details follow:

For initial condition $Y(0)$, we have shown that the solution is of the form

$$Y(t) = Z(t) \cdot Y(0)$$
$$= P(t) \cdot e^{tB} \cdot Y(0)$$

Now ~~we~~ recall that $P(0) = I$, identity.

$$\Rightarrow \quad Y(t) = P(t)\, e^{tB} \; P(0)^{-1}\, Y(0).$$

$$P(t)^{-1} Y(t) = e^{tB} \; P(0)^{-1} Y(0).$$

Say $U = P(t)^{-1} Y(t)$. Then

$$\boxed{U(t) = e^{tB}\; U(0)}$$

But we know that this $U(t)$ is the solution to the time-invariant ODE $\dot{U} = BU$ with B = constant matrix & $U(0)$ - initial conditions

Thus $U = P(t)^{-1} Y$ is a time-periodic coordinate transformation that transforms the time-periodic system $\dot{Y} = J(t)Y$ to a time-invariant $\dot{U} = BU$.

Note, however that there is no guarantee that B
or P(t) is real.

Therefore, this transformation may result in
"complex" matrix dynamics.

Is there a REAL transformation which has
time-invariant dynamics? YES, but for this
we have to consider two-periods of the motion rather
than just one, as shown next.

(2) A real time-periodic transformation that takes the time-periodic dynamics to time-invariant dynamics

We have noted there exists B such that $e^{TB} = C$ with B possibly complex.

Lemma: If $e^{TB} = C$, then $e^{T\bar{B}} = C$ where $\bar{B}$ is the complex conjugate of B.

Pf. $e^{TB} = C$. $\quad \bar{C} = C$ because C is real.

$$\overline{e^{TB}} = \bar{C} = C$$

$$\Rightarrow \boxed{e^{T\bar{B}} = C}$$

Thus, $C^2 = C.C = e^{TB} . e^{T\bar{B}} = e^{T(B+\bar{B})}$.

$B+\bar{B}$ is real.. $= 2Q$, say

So, there exists a real matrix Q such that $e^{2TQ} = C^2$.

Recall that we have

$$Z(t+T) = Z(t)\, e^{TB} = Z(t)\, C.$$

$$\Rightarrow Z(t+2T) = Z(t)\, C^2 = Z(t) . e^{2TQ}.$$

<u>Lemma</u> : $z(t+2T) = z(t) \cdot e^{2TQ}$ has a solution of the form $z(t) = S(t)\, e^{tQ}$.

<u>Pf</u> Let plug the proposed solution in the equation

$$z(t+2T) = S(t+2T)\, e^{tQ+2TQ}$$
$$= z(t)\, e^{2TQ} = S(t)\, e^{tQ} \cdot e^{2TQ}$$

$$\Rightarrow S(t+2T) = S(t)$$

Thus $z(t) = S(t)\, e^{tQ}$ is a solution of $z(t+2T) = z(t)\, e^{2TQ}$ if $S(t) = S(t+2T)$ is a $2T$-periodic function.

Thus, for a given initial conditions $y(0)$, the solution can be written in the form

$$y(t) = Z(t)\, y(0)$$
$$= S(t)\, e^{tQ}\, y(0)$$

$$\Rightarrow S(t)^{-1} y(t) = e^{tQ}\, y(0) = e^{tQ}\, S(0)^{-1}\, y(0)$$ because $S(0) = I$.

Thus if we define new variable $\boxed{V(t) = S(t)^{-1} y(t)}$

it satisfies the equation $\boxed{V(t) = e^{tQ}\, v(0)}$ which is the solution

of <u>Real</u> <u>time-invariant</u> ODE $\boxed{\dot{V} = QV}$

where Q is a real <u>matrix</u>. $S(t)$ is a time-periodic transformation of variables.

- Note that we have not described how to find $S(t)$ but can be found <u>easily</u> by numerical simulation.

(Exercise: describe how you would find Q and $S(t)$).

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