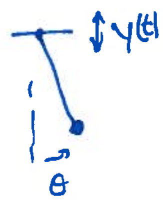
Examples of time-periodic systems.

Mathieu's equation.

Mathieu's equation is a <u>linear</u> time-periodic second-order ODE which arises naturally as linearizations of various systems.

<u>Examples of nonlinear systems that give rise to similar equations</u>:

→ 

pendulum on a vertically excited pivot.

→ 

pendulum whose length is changed periodically.

<u>Exercise</u>:

- Write the nonlinear equations for the above pendulum system for $\theta(t)$ – second order ODE for $\theta(t)$. in terms of $y(t)$ or $l(t)$.
- Linearize the nonlinear ODEs about $(\theta = 0, \dot{\theta} = 0)$ which is a fixed point. Also about $(\theta = \pi, \dot{\theta} = 0)$.
- The resulting equations will have time-periodic aspects.

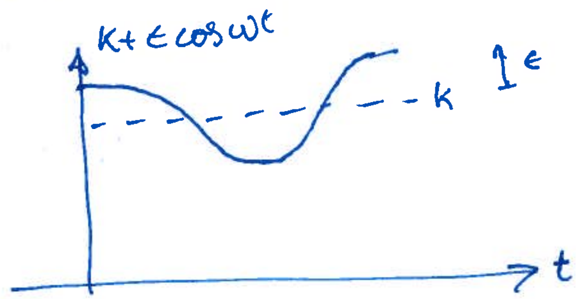
Mathieu's equation is

$$\ddot{x} + c\dot{x} + (k + \epsilon \cos \omega t)\, x = 0$$

ω, c, k, ϵ are constants

Thus, this is an equation with a time-periodic stiffness.

$k + \epsilon \cos \omega t$.



Fixed point $(0,0)$. $x^* = 0$ and $\dot{x}^* = 0$. Only fixed point.

When $\epsilon = 0$, we know that $(0,0)$ is stable asymptotically if $k > 0$ and $c > 0$.

and unstable when $k < 0$ or $c < 0$.

When $\epsilon > 0$, the results are no longer immediately obvious.

eg. It is possible to be unstable with $c > 0$ and $k > 0$ and $k \gg \epsilon$ so that $k + \epsilon \cos \omega t > 0$ for all time!

That is, having a large positive mean stiffness says nothing about stability.

eg 2: It is possible to have $k<0$ and $\epsilon>0$, such that $(0,0)$ is stable!! That is, the mean stiffness k can be negative and still have stability)

(This is actually the regime at which an inverted pendulum $(\theta=\pi)$ is stabilized by support oscillations of sufficient magnitude.

Parameter sweep to study stability.

We can evaluate whether $(0,0)$ will be stable by using Floquet theory. We rewrite the 2^{nd} order ODE in the standard first order form.

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \text{ say.}$$

Then
$$\dot{Y} = \begin{bmatrix} 0 & 1 \\ -(k+\epsilon\cos\omega t) & -c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

This equation is of the form $\dot{Y} = J(t)\,Y$ with J being time-periodic with period $T = \frac{2\pi}{\omega}$.

To evaluate stability, we construct the C matrix, called the "monodromy matrix", and check if all(abs(eig(C))) < 1.

<u>Two ways of evaluating C</u>

$\boxed{\text{Method (1)}}$ Start with initial condition $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, integrate ODE for time T. to get $\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$.

Start with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, integrate ODE for time $T = \frac{2\pi}{\omega}$ to get $\begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix}$

(This works ONLY if we have LINEAR ODE's of the form $\dot{Y} = J(t)\,Y$.

$\boxed{\text{Method 2}}$

→ Write a function H that takes as input initial condition $Y(0)$ and gives as output $Y(T)$ obtained by integrating the ODE for time $T = \frac{2\pi}{\omega}$.

→ C = Jacobian of H with respect to $Y(0)$ about $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
a 2x2 matrix

<u>Note</u>: Be careful to use <u>appropriate</u> finite difference step size (not too small/large)

- Notes: Both methods are applicable to general N-dimensional systems.

  - Method 2 is easily generalized to nonlinear systems with or without time-dependence, where the linearization can be about a fixed point or a periodic motion.

Ok, see uploaded MATLAB program that fixed parameters c and ω, and swept through different values of k and ϵ, to see for what values of k and ϵ the system was stable.

→ We used Method 2. We check if

$$\text{all}(\text{abs}(\text{eig}(C))) < 1 \Rightarrow \text{stable}$$

else unstable.

→ We use $\omega = 1$. $c =$ two different values, one small like 0.001 and one larger, like 0.1.

The parameter-sweep stability plots are attached in the next few pages.

In the computer-generated figures: (next 2 pages).

- We see that the parameter regions in k-ϵ space where $(0, 0)$ is stable is quite complex in shape.

- There are "tongues of instability" even for high positive values of mean stiffness k.

- We see that increasing the value of damping reduces the size of k-ϵ regions corresponding to instability (pushes the regions upward).

- We see that there is a stable region even for negative mean stiffness k.

(As noted earlier, this is the relevant region that stabilizes the inverted pendulum with vertical pivot oscillations).

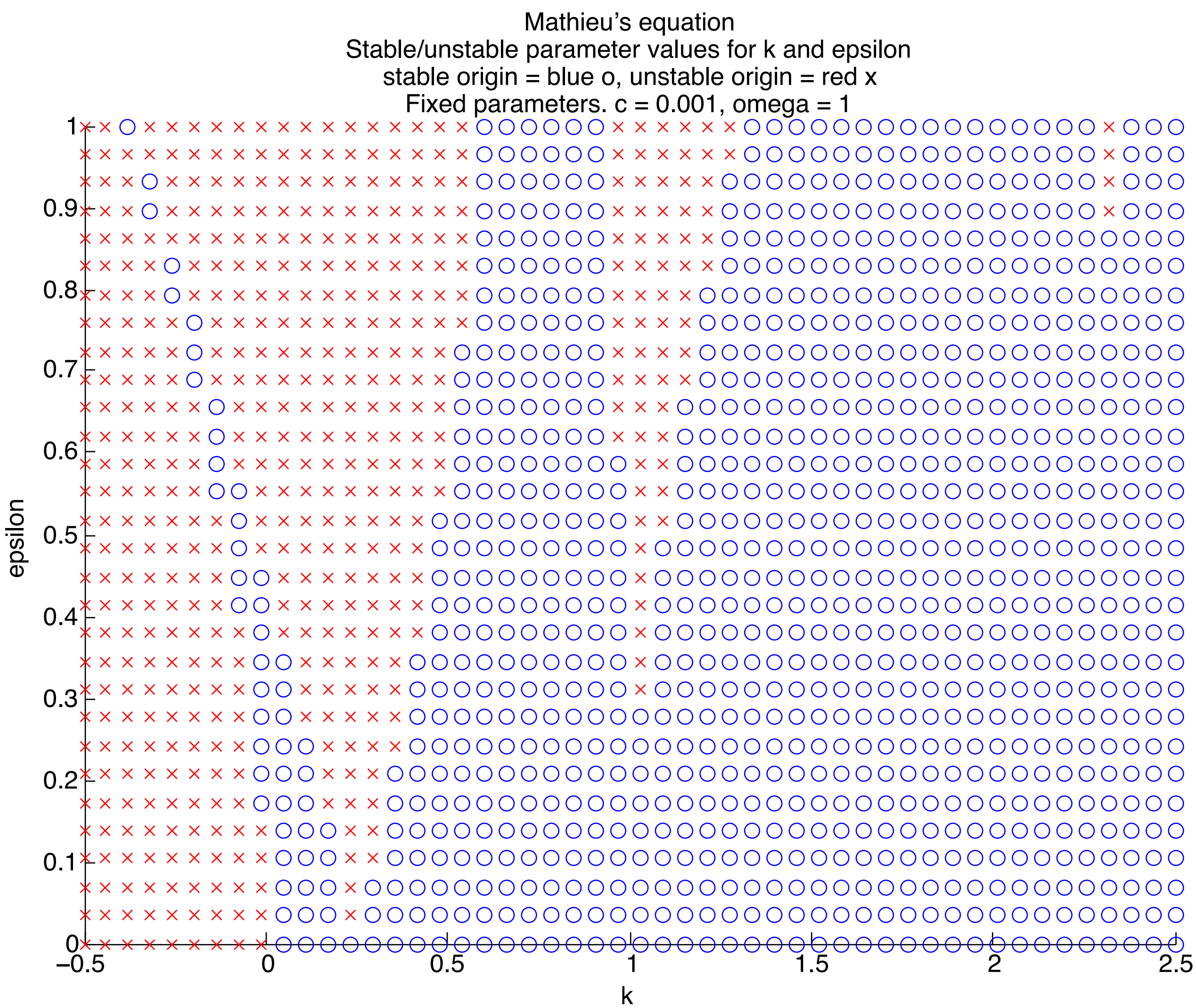
Note 1: Mathieu's equation is also considered an example of "parametric excitation", because the system is excited by changing a parameter, namely the stiffness.

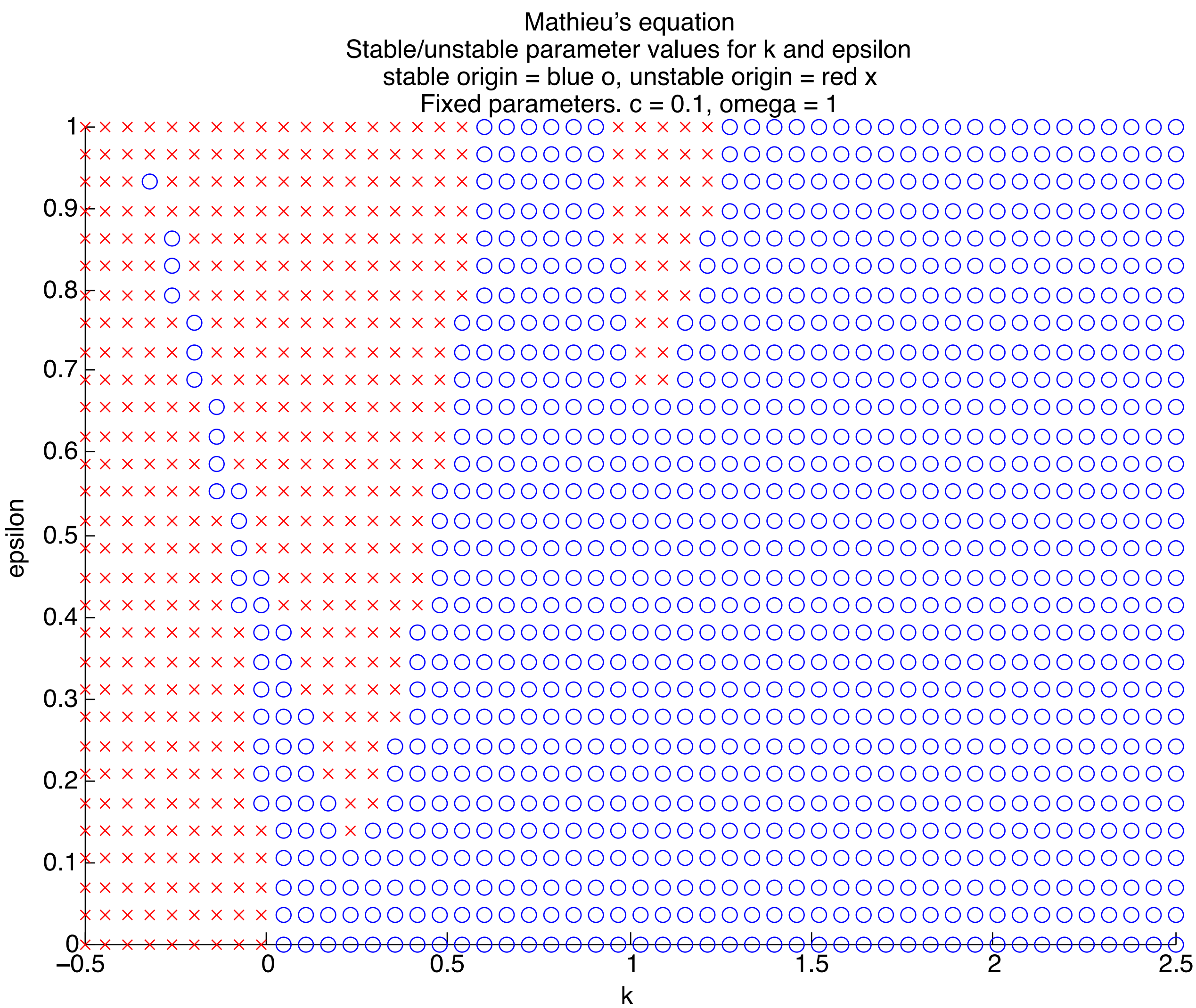
Note 2 : We will see how to think about the mathieu's equations analytically later in the course.

Note 3 : A slightly more general 2^{nd} order equation related to the Mathieu's equation is called the "Hill's equation", that also arises in

$$\frac{d^2y}{dt^2} + \underbrace{f(t)}_{\text{some periodic function}} y = 0 .$$

perhaps with damping as well.


Mathieu's equation
Stable/unstable parameter values for k and epsilon
stable origin = blue o, unstable origin = red x
Fixed parameters. c = 0.001, omega = 1
epsilon
k
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.5
0
0.5
1
1.5
2
2.5



Mathieu's equation
Stable/unstable parameter values for k and epsilon
stable origin = blue o, unstable origin = red x
Fixed parameters. c = 0.1, omega = 1
epsilon
k
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
−0.5
0
0.5
1
1.5
2
2.5

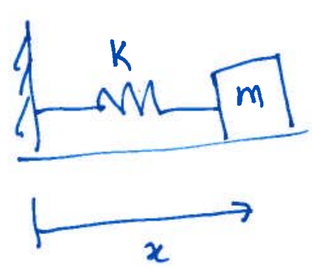
Changing stiffness can cause instability even if the stiffnesses are very positive

- We have already seen the Mathieu's equation example.
- Here are a couple of more examples in which the stiffness changes are state-dependent, rather than time-dependent

- State-dependent stiffness changes can occur, for instance, in gears - depending on whether one or two gear teeth are in contact with another gear's teeth, the stiffness mediating the two gears.

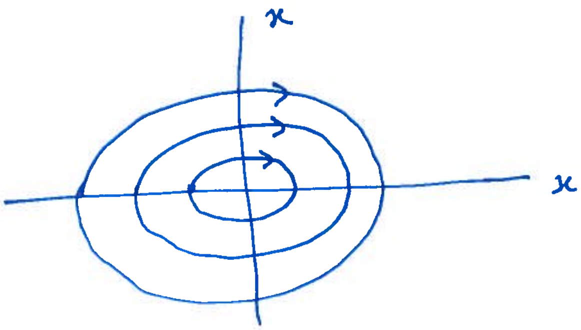
- Simple example 1.

Consider a simple harmonic oscillator with no damping



$$m\ddot{x} + kx = 0.$$

Recall that the phase portrait for this system consists of a family of periodic orbits.

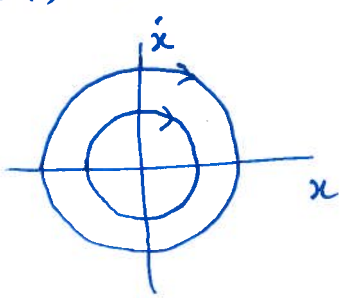


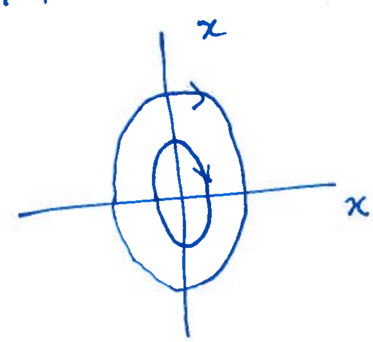
Recall that the periodic orbits are ellipses with equation given by the energy conservation equation

$$\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = E.$$

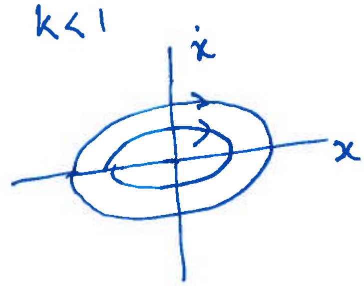
Say $m=1$. $\Rightarrow$ $x^2 + \frac{\dot{x}^2}{k} = C$.

if $k=1$, we have circles, if $k>1$





if $k<1$



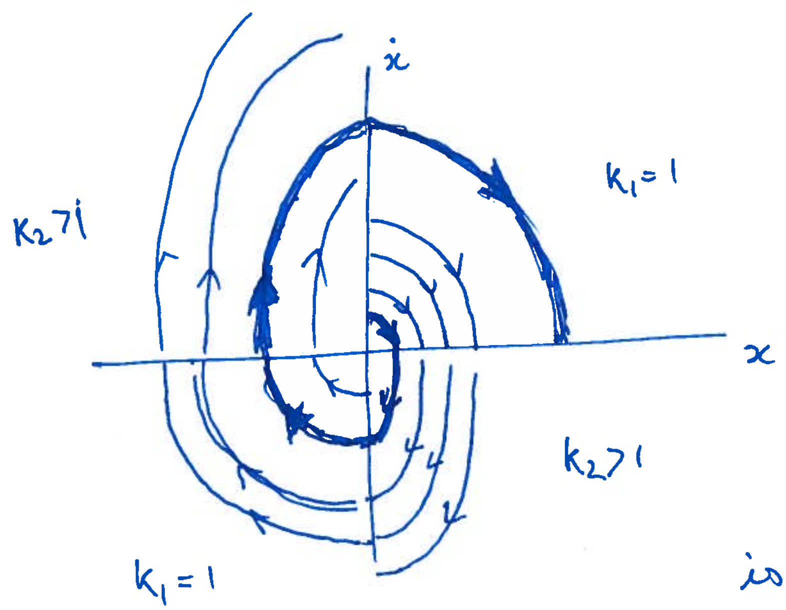
Now, we switch the stiffness between 2 levels, so as to make $(0,0)$ (1) unstable

(2) stable.

Say the spring stiffness k switches between

k_1 and k_2.

with $k_1 = 1$ and $k_2 > 1$.



From $k_1 \to k_2$ when $\dot{x} = 0$.

From $k_2 \to k_1$ when $x = 0$.

As we can see, $(0,0)$ is unstable for this system.

Physical intuition: We switch the stiffness from low (k_1) to high (k_2) when the displacement $|x|$ is maximum. ie, the energy in the system goes from

$\frac{1}{2} k_1 x_{max}^2$ to $\frac{1}{2} k_2 x_{max}^2$.

Then, we switch stiffness from high to low (k_2 to k_1) when displacement $= 0$. There is no change in energy because of this switch.

Thus, we are constantly adding energy.

Hence unstable.

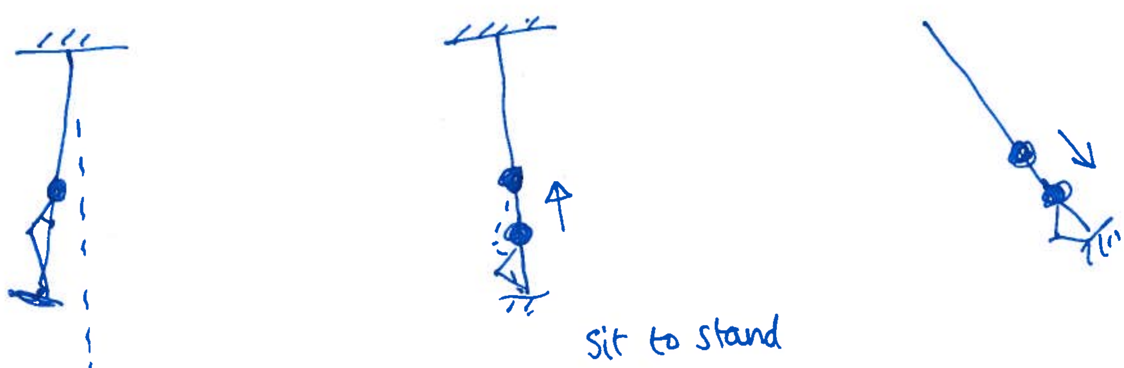
Note 1: We can also make (0,0) asymptotically stable by appropriate switching of stiffnesses.

Exercise: Devise such appropriate stiffness changing rules and draw the appropriate phase portraits.

Note 2: The above systems are non-smooth as the differential equation discontinuously depending on state. They are also hybrid systems.

Yet another example. Pumping a swing by appropriately timed changes between sitting and standing.

(see paper by Wirkus, Rand, and Ruina).



Sit to stand when $\theta = 0$.

($\Rightarrow$ Increase in KE).

Stand to sit when $\theta = \theta_{max}$ & $\dot{\theta} = 0$.

(decrease in PE but smaller than the increase in KE)