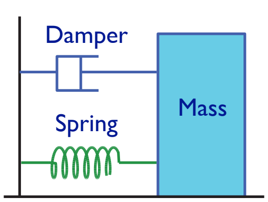
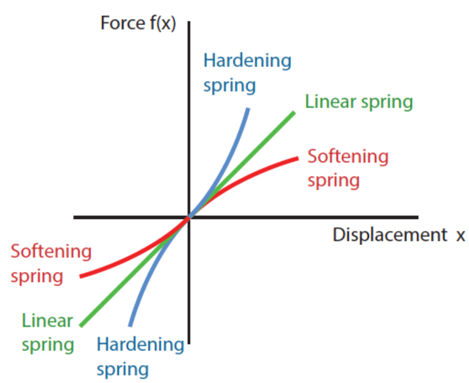
Basic phenomenology of simple nonlinear vibration

(free and forced)

Manoj Srinivasan (2016)

Nonlinear spring-mass system





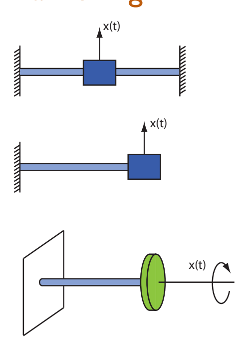
$$\ddot{x} + kx + \alpha x^3 = 0$$

No damping

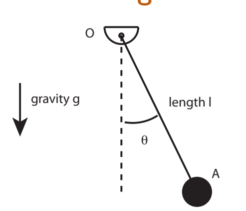
Hardening or stiffening spring: $\alpha > 0$

Softening spring: $\alpha < 0$

Hardening

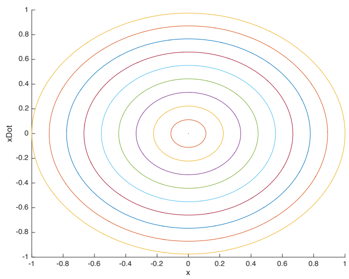


Softening



Nonlinear spring-mass system

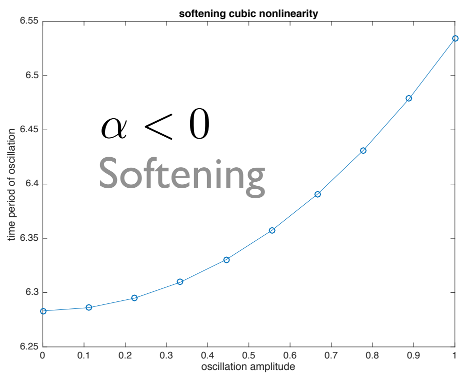
$$\ddot{x} + kx + \alpha x^3 = 0$$ No damping



Frequency (time period) of free vibration oscillations depends on oscillation amplitude

unlike for linear spring-mass system

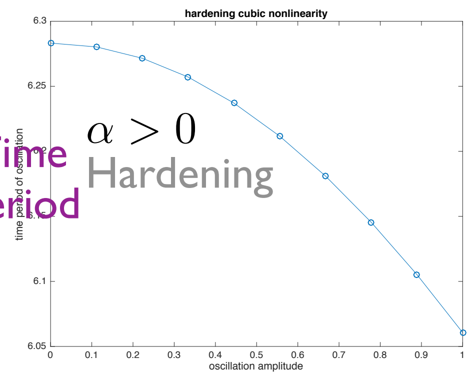
Time period



$\alpha < 0$ Softening

Oscillation amplitude

Time period



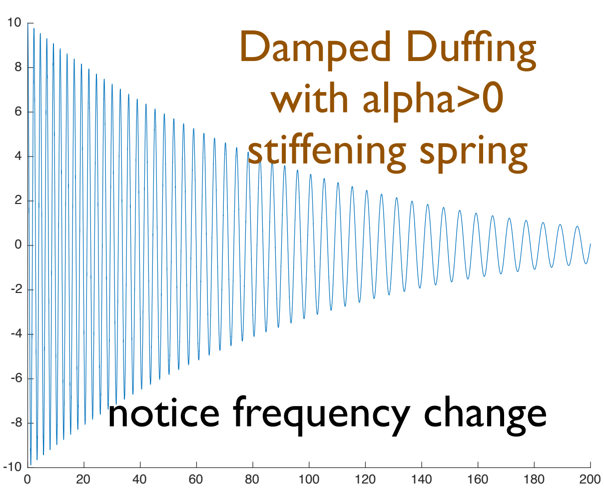
$\alpha > 0$ Hardening

Oscillation amplitude

Damped free vibrations of Duffing equation

how would you expect the frequency to change as the oscillation amplitude decreases?

x(t)

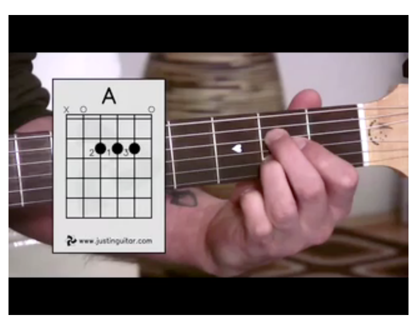


Damped Duffing with alpha>0 stiffening spring

notice frequency change

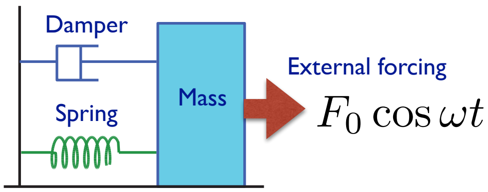
t

guitar "pitch glide" due to amplitude dependence of frequency? (assume stiffening)



https://www.youtube.com/watch?v=VkkOFLouQDs

Linear spring-mass-damper system



$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 \left(\frac{F_0}{k}\right) \cos \omega t$$

Damping ratio $\zeta = \dfrac{c}{2m\omega_n}$

Natural frequency $\omega_n = \dfrac{k}{m}$

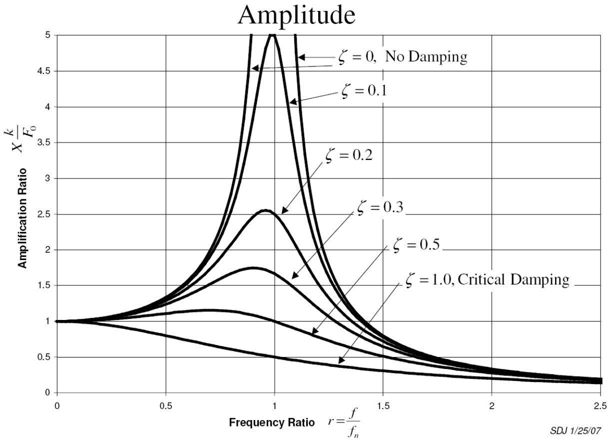
Normalized force amplitude (units of length) = $A = \dfrac{F_0}{k}$ (Static deflection to static force F_0)

Frequency response of linear spring-mass-damper system

$$\text{Amplification ratio} = \frac{\text{Amplitude of 'steady' response } x(t)}{\text{Forcing amplitude normalized by stiffness}} = \frac{X}{F_0/k}$$

$$\text{Phase angle } \phi = \text{Response phase} - \text{forcing phase}$$

Amplitude



Phase

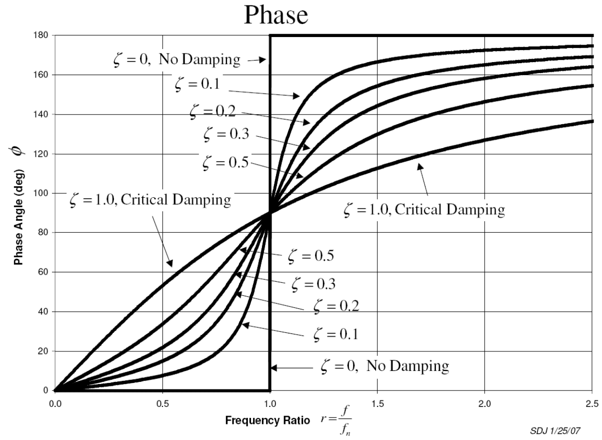
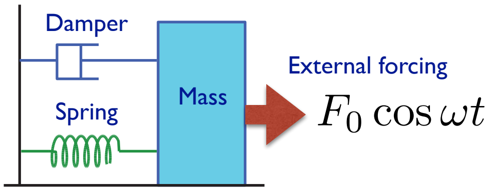
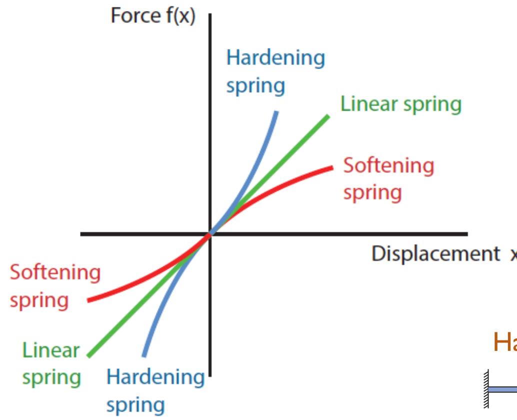
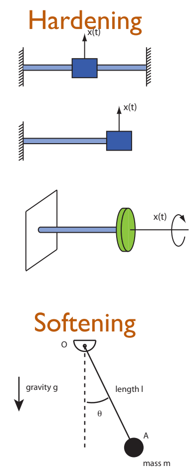


figure source: Wikipedia

Duffing equation with forcing







$$m\ddot{x} + c\dot{x} + kx + \boxed{\alpha x^3} = F_0 \cos \omega t$$

cubic nonlinearity

Hardening or stiffening spring: $\alpha > 0$

Softening spring: $\alpha < 0$

Cubic nonlinearity with or without quadratic nonlinearity

When we do Taylor series of an odd function about an equilibrium for with spring force = 0

Duffing's equation $\quad m\ddot{x} + c\dot{x} + kx + \boxed{\alpha x^3} = F_0 \cos \omega t$

cubic

When we do Taylor series of odd function about an equilibrium with spring force not = 0

(Recall HW1 for illustrative example)

$$m\ddot{x} + c\dot{x} + kx + \boxed{\alpha x^3} + \boxed{\beta x^2} = F_0 \cos \omega t$$

cubic quadratic

(or) just Taylor series of a not-odd function

Frequency response of Duffing equation (cubic nonlinearity)

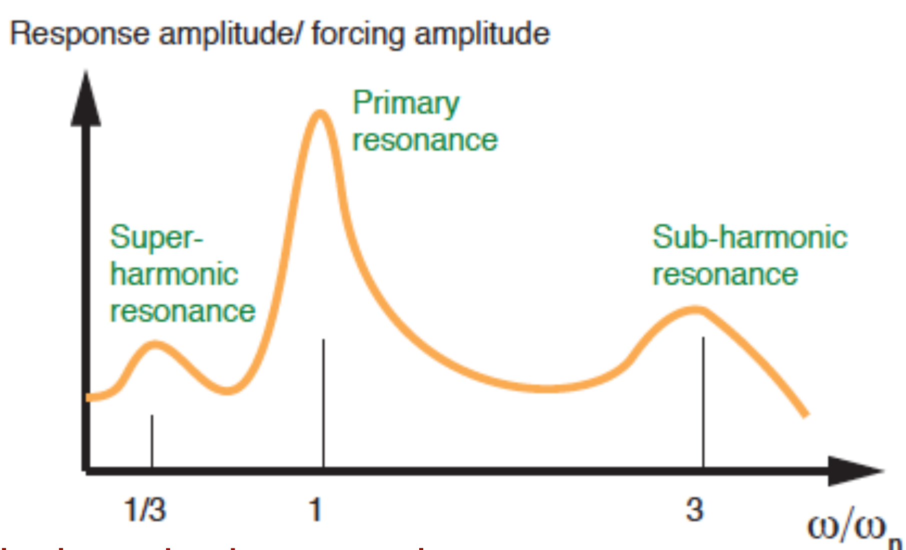
Primary resonance .
Big response amplitude when
forcing frequency ω_o ~ 'linear' natural frequency ω_n

Secondary resonance

Super-harmonic resonance: Big response amplitude when
forcing frequency $\omega_o \sim \omega_n$ / integer (example: $\omega_o \sim \omega_n/3$)

Sub-harmonic resonance: Big response amplitude when
forcing frequency $\omega_o \sim \omega_n$ x integer. (example: $\omega_o \sim 3\omega_n$)

Primary resonance and secondary resonances



Multiple peaks due to nonlinearity
even though it is a single DOF system

(Recall that an N DOF linear system (N>1) will have multiple peaks due to there being N modes and corresponding natural frequencies)

Linear vs Hardening vs Softening

How the primary resonance's frequency response change

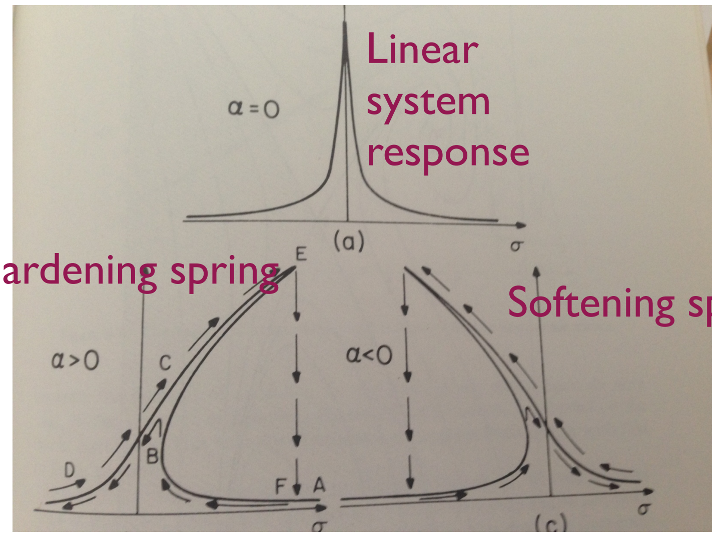
Response amplitude (y axis) vs forcing frequency (x axis)
for 3 cases

Nayfeh and Mook

Linear system response

Hardening spring

Softening spring



forcing frequency   forcing frequency

arrows indicate response obtained in forward and backward sweeps

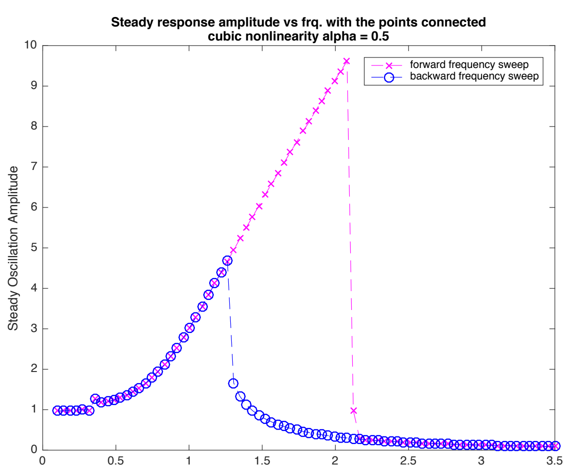
Hardening vs Softening

Amplitude response by simulation until steady state

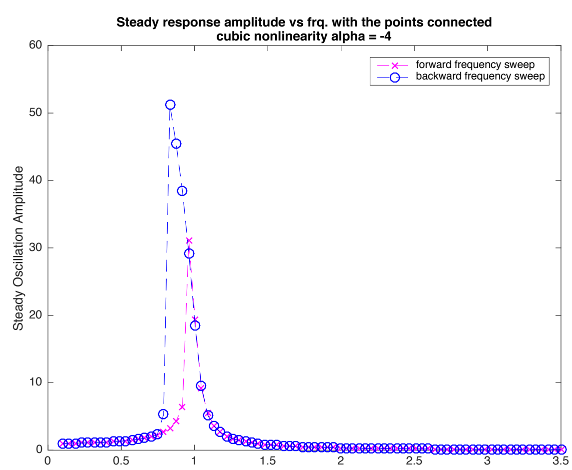
see MATLAB program

Forward sweep (magenta) and Backward sweep (blue) shown

Hardening



Softening



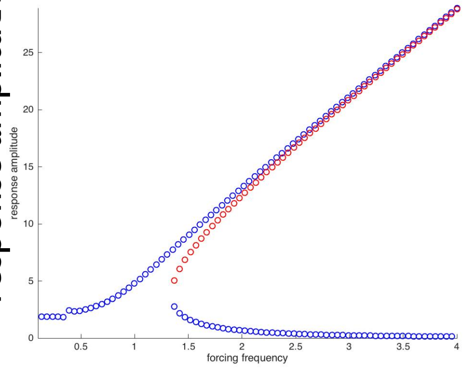
Amplitude response obtained by finding fixed points of Poincare maps (so we can find both stable and unstable motions)

see MATLAB program

Frequency response for forced Duffing with hardening spring

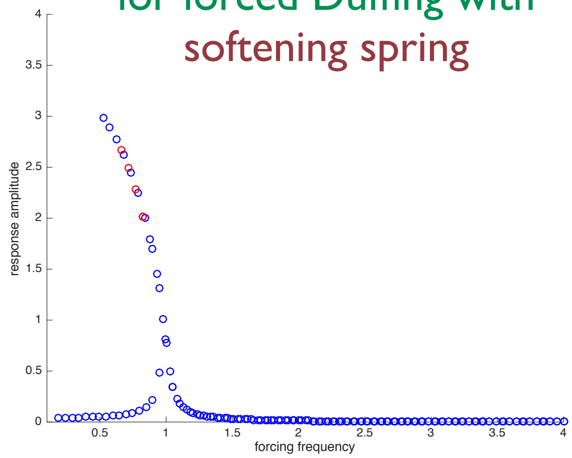
Frequency response for forced Duffing with softening spring

response amplitude



forcing frequency

response amplitude

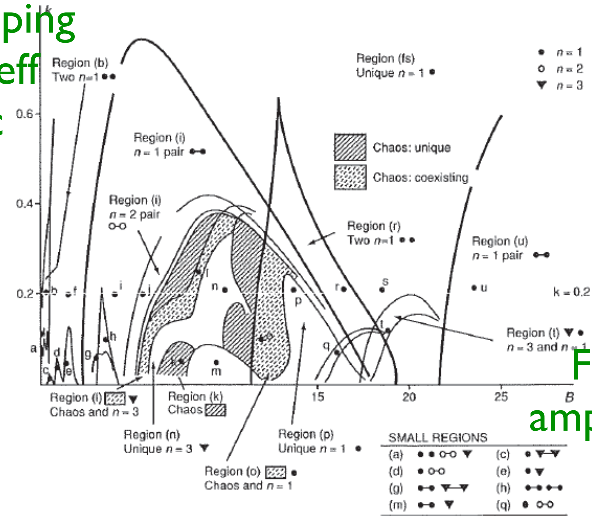


forcing frequency

blue = stable periodic response
red = unstable periodic response

Ueda shows that the fully nonlinear forced Duffing (linear stiffness = 0) has many parameter regimes with many different behavior

Damping coeff c



Forcing amplitude A

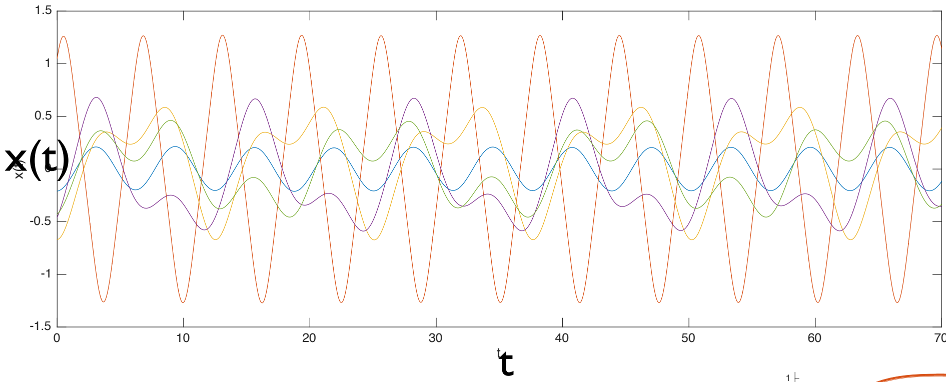
Figure 1.10 Twenty-one major regions (a) to (u) of the various long-term behaviours of Duffing's equation ($\ddot{x} + k\dot{x} + x^3 = B\cos t$) as mapped by Ueda, as a function of damping magnitude k and forcing amplitude B. Reproduced, with permission of SIAM, from Ueda (1980a)

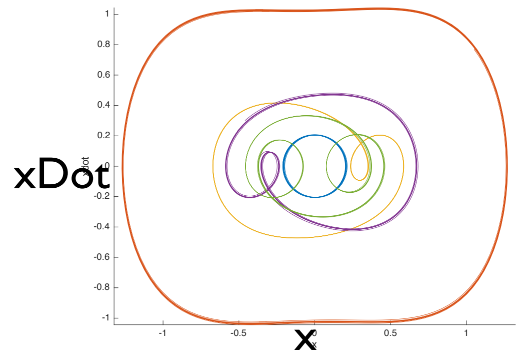
See paper uploaded, Ueda 1991.

Following slides show some MATLAB illustrations of this complexity

Five different co-existing periodic motions

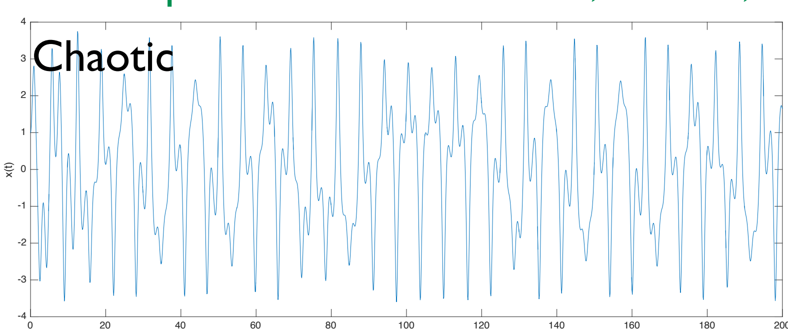
For the same parameter values: k = 0, A = 0.2, c = 0.08, omega = 1

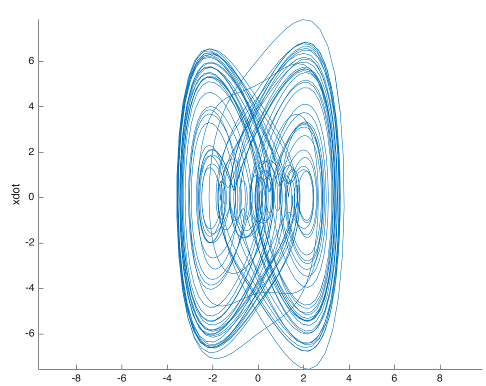




Chaotic motion (unique)

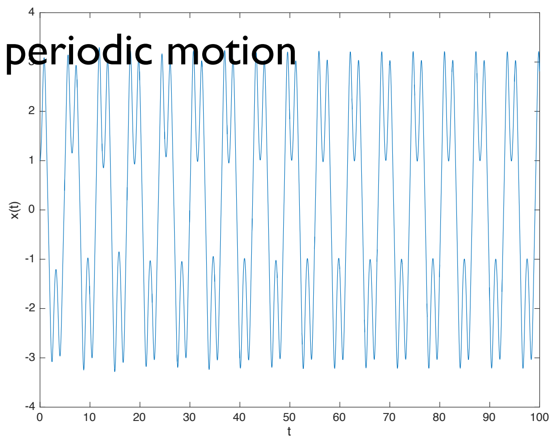
For the same parameter values: k = 0, A = 10, c = 0.1, omega = 1

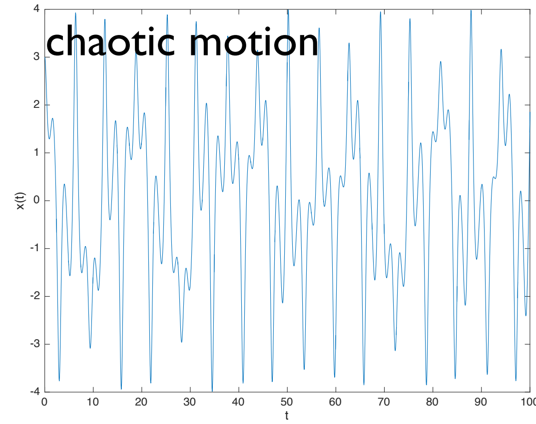


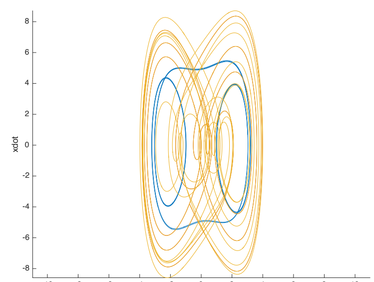


Co-existing periodic motion and chaotic motion

For the same parameter values: k = 0, A = 12.5, c = 0.1, omega = 1

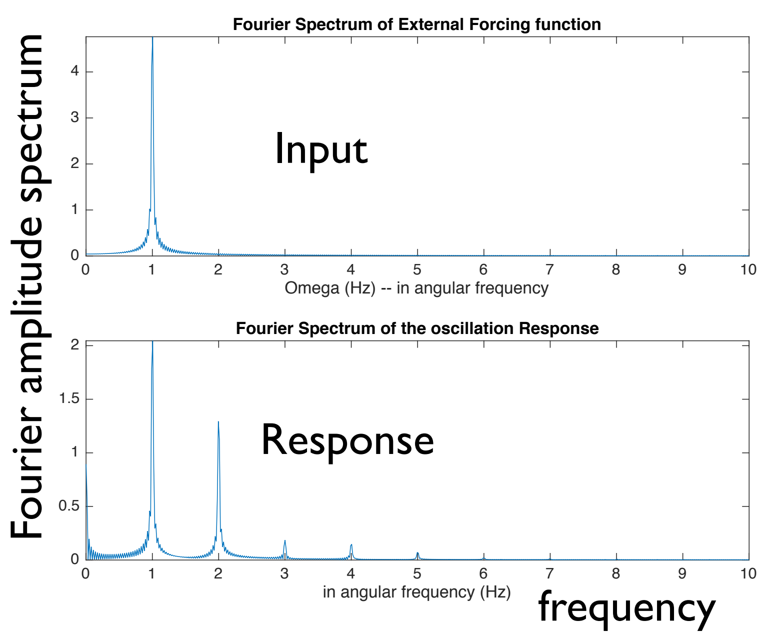






Frequency content of response

Primary resonance (forcing ω = 1)

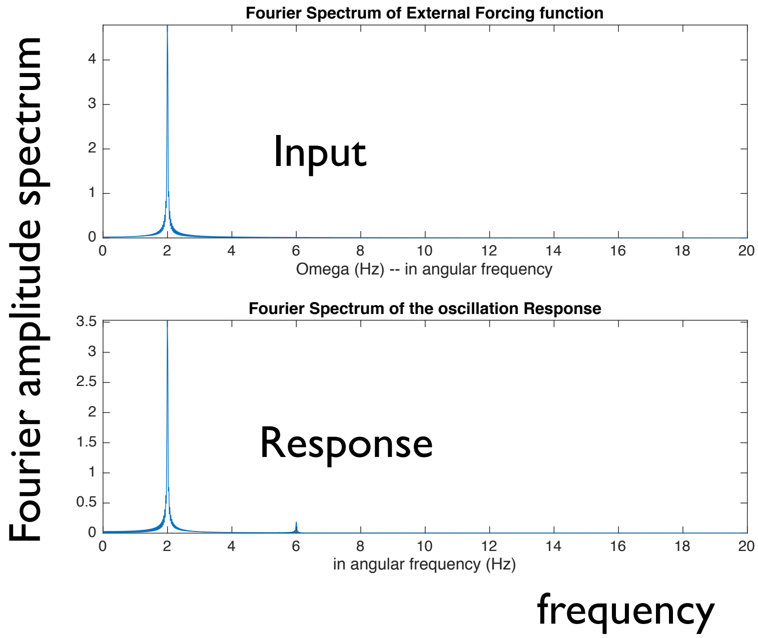


c = 0.1
m = 1, k = 1
epsilon = 0.4
omega = 1
A = 5

Even when the response has the same frequency as the forcing, there can be other harmonics in response due to nonlinearity (unlike linear systems)

Frequency content of response

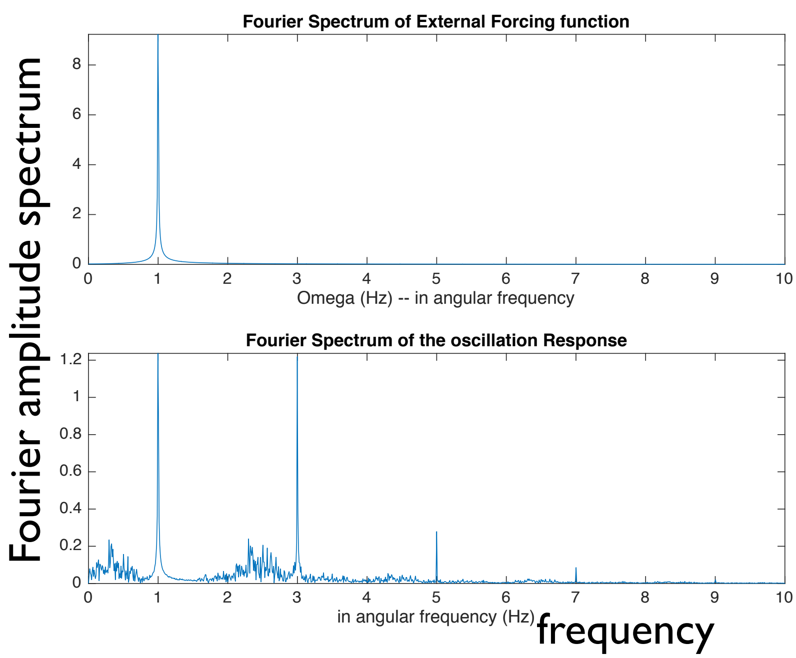
Forcing freq $\omega = 2$

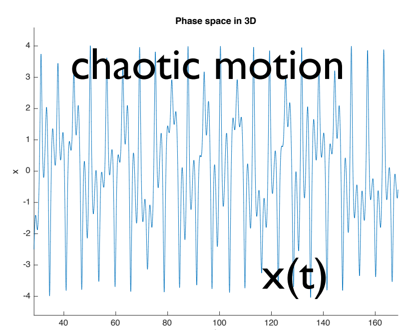


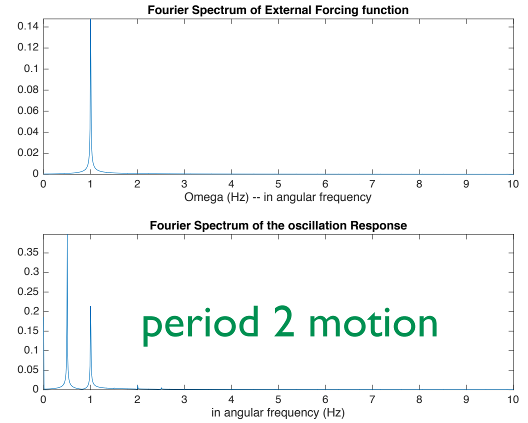
c = 0.1
m = 1, k = 1
epsilon = 0.4
omega = 2
A = 5

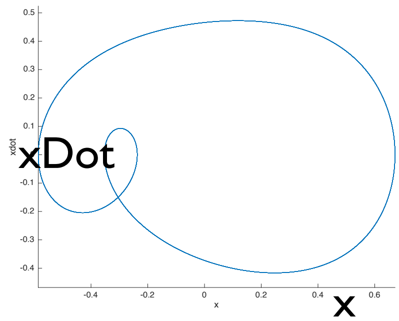
far from resonance we don't see much higher harmonics in response

Chaos = broad band frequency content



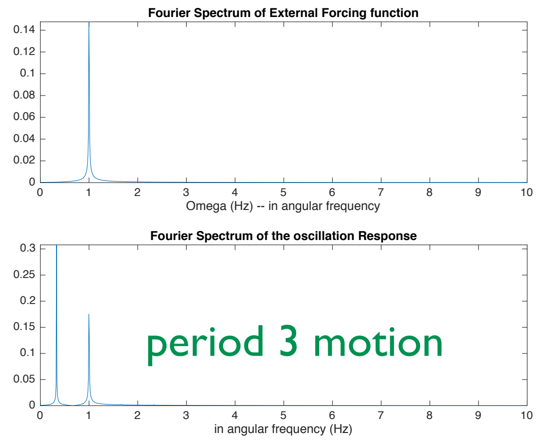


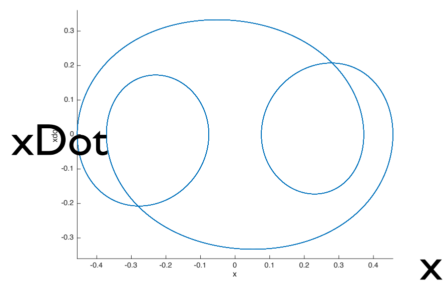




Ueda
(Manoj
notation)

c = 0.08
k = 0
epsilon = 1;
omega = 1
A = 0.2;





Beware:

softening cubic nonlinearity
and purely quadratic nonlinearity

have regimes where the stiffness is 'negative' for some large amplitudes => system can go to infinity if the forcing is not small enough

Fix:

A quadratic nonlinearity could be accompanied by a stiffening cubic nonlinearity which keeps the motion bounded

A softening cubic nonlinearity could be accompanied by a stiffening x^5 nonlinearity, which keeps the motion bounded