Perturbation theory

Example 1 (Bender & Orszag).

Roots of a cubic polynomial.

$$x^3 - 4.001x + 0.002 = 0 \qquad -(1)$$

Step 1a Introduce a small parameter ϵ.

$4.001 = 4 + \epsilon$ where $\epsilon = 0.001$.

$0.002 = 2\epsilon$.

$\Rightarrow$ (1) becomes $x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad -(2)$

Step 1b. Make sure $\epsilon = 0$ is easy to solve.

$$x^3 - 4x = 0 = x(x^2-4)$$

$\Rightarrow$ 3 solutions $x^* = 0, 2, -2$.

Step 2 Let us try to find the solutions when $\epsilon \neq 0$

Say we want to find the solution near $x^* = -2$. when $\epsilon = 0$

Assume that when $\epsilon \neq 2$, $x^*(\epsilon) = -2 + a_1\epsilon + a_2\epsilon^2 + \ldots \quad (3)$

$\boxed{\text{a series expansion in } \epsilon}$

Substitute 3 in 2.

$$\left(-2+a_1\epsilon+a_2\epsilon^2\right)^3-(4+\epsilon)\left(-2+a_1\epsilon+a_2\epsilon^2\right)+2\epsilon=0 \quad (4)$$

ignoring higher order terms

Let us expand 4 but keep only terms until $O(\epsilon^2)$
[we can use MATLAB's symbolic toolbox]

$$\left[\begin{aligned}&-\cancel{8}+\left(-6a_1^2\epsilon^2\right)+12a_1\epsilon+12a_2\epsilon^2\\&\quad+2\epsilon-4a_1\epsilon\mp a_1\epsilon^2-4a_2\epsilon^2+\cancel{8}\\&\qquad+2\epsilon\end{aligned}\right]=0$$

$$+O(\epsilon^3).$$

Collect terms containing ϵ, ϵ^2, etc.

$$\epsilon\left[12a_1+2-4a_1+2\right]+\epsilon^2\left[-6a_1^2+12a_2\mp a_1-4a_2\right]=0$$

Because we want the solution to be valid for all small ϵ we set the individual coefficients of powers of ϵ to zero.

$$\Rightarrow 12a_1-4a_1+4=0$$

$$8a_1+4=0$$

$$\boxed{a_1=-1/2}$$

$$-6a_1^2+12a_2+a_1-4a_2=0$$

$$8a_2=6a_1^2-a_1=6\left(\frac{-1}{2}\right)^2+\frac{1}{2}$$

$$\Rightarrow \boxed{a_2=1/8}$$

Thus, the "perturbation solution" is

$$x^* = -2 - \frac{\epsilon}{2} + \frac{1}{8}\epsilon^2 + O(\epsilon^3).$$

This is the solution near the -2 root.

— Using a similar procedure, we can show that the solution near the $+2$ root

is $x^* = +2 + 0\,\epsilon + 0\,\epsilon^2 + 0\,\epsilon^3 + \ldots$

i.e., $x^* = 2$ is an exact solution of the original equation for all ϵ!

— The solution near the 0 root is

$$x^* = 0 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3).$$

Remark: This is an example of "regular perturbation" in which the $\epsilon = 0$ is well-defined and the $\epsilon \neq 0$ grow continuously out of the $\epsilon = 0$ solution.

When this is NOT the case, the problem is considered "singular perturbation".

Duffing equation FREE VIBRATION

$$\ddot{x} + x + \underbrace{\epsilon x^3}_{\text{cubic nonlinearity}} = 0. \quad (1)$$

(1) Regular perturbation expansion (will lead to "Secular" terms) — not very convenient.

(2) Lindstedt - Poincare method,

(3) Method of multiple scales

(4) Averaging.

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(1) <u>Regular perturbation expansion</u>

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \quad (2)$$

$$\ddot{x} = \ddot{x}_0(t) + \epsilon \ddot{x}_1 + \epsilon^2 \ddot{x}_2 + \cdots \quad (3)$$

Substitute (2) & (3) in (1). ignore terms $O(\epsilon^2)$

$$\ddot{x}_0 + \epsilon \ddot{x}_1 + x_0 + \epsilon x_1 + \epsilon (x_0 + \epsilon x_1)^3 = 0 \quad (4)$$

Collect constant terms, ϵ terms, and ignore $O(\epsilon^2)$ terms

$$[\ddot{x}_0 + x_0] + \epsilon [\ddot{x}_1 + x_1 + x_0^3] + \underbrace{O(\epsilon^2)}_{\simeq 0} \quad (5)$$

Set coefficients of powers of ϵ to zero.

$$\ddot{x}_0 + x_0 = 0 \qquad (6)$$

$$\ddot{x}_1 + x_1 + x_0^3 = 0 \qquad (7).$$

Solving (6) is easy.

$$x_0 = A_0 \cos(\omega t - \phi) \qquad (8) \quad , \quad \omega = 1$$

Substitute in (7).

$$\ddot{x}_1 + x_1 + A_0^3 \cos^3(\omega t - \phi) = 0.$$

Recall $\cos^3\theta =$

$$= \frac{\cos 3\theta + 3\cos\theta}{4}$$

$$\Rightarrow \underbrace{\ddot{x}_1 + x_1}_{\substack{\omega_n = 1 \\ \text{natural} \\ \text{frequency}}} + \frac{A_0^3}{4}\cos(3\omega t - \phi) + \underbrace{\frac{A_0^3\, 3\cos(t - \phi)}{4}}_{\text{forcing frequency } \omega = 1} = 0$$

resonance

this term is called $\boxed{\text{a "Secular term"}}$.

$x_1(t) =$ 

Using this technique, we can show that all $X_n(t)$ for $n \geqslant 2$ give ∞ at $t \to \infty$.

Each term in the series

$$\dot{X} = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots$$ goes to infinity.

We can show that the sum is finite at every t and as $t \to \infty$. (see Bender & Orszag).

But this is cumbersome. There are other methods that do not produce these infinities.

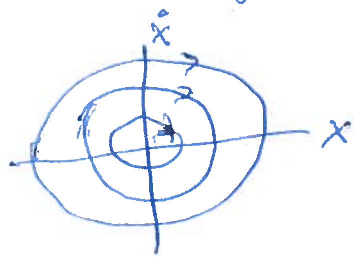
(2) Lindstedt - Poincaré method

Circumvents the secular terms by introducing additional unknowns in the perturbation expansion and picking those unknown coefficients in a manner that the secular terms vanish

Same equation. Duffing with no damping.

$$\ddot{x} + x + \epsilon x^3 = 0 \quad (1)$$

We are seeking free vibration solutions. (The solutions are bounded & periodic)



(We already know that the oscillation period changes with amplitude).

Introduce a new time-like variable τ

$$\boxed{\tau = \omega t} \quad (2)$$

where $\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots$ (3) with $\omega_0 = 1$

As we will see, this will allow us ~~to~~ ~~obtain~~ obtain a amplitude-frequency relation.

Rewrite (1) using (2)

$$\frac{d^2x}{dt^2} = \omega^2 \frac{d^2x}{d\tau^2} \Rightarrow \omega^2 \frac{d^2x}{d\tau^2} + x + \epsilon x^3 = 0 \quad (4)$$

Further, we assume that (as before)

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 \quad \ldots \qquad (5)$$

all x's being dependent on τ. $x(\tau)$, $x_0(\tau)$ etc.

Substitute (5) in (4), and keep only terms until $O(\epsilon)$.

$$(\omega_0 + \omega_1 \epsilon)^2 \left(\frac{d^2 x_0}{d\tau^2} + \epsilon \frac{d^2 x_1}{d\tau^2} \right) + (x_0 + \epsilon x_1) + \epsilon (x_0 + \epsilon x_1)^3 = 0 + O(\epsilon^2)$$

Expand and collect terms of ϵ^0, ϵ^1, etc.

$$\left[\frac{d^2 x_0}{d\tau^2} + x_0 \right] + \epsilon^1 \left[\frac{d^2 x_1}{d\tau^2} + x_1 + 2\omega_1 \frac{d^2 x_0}{d\tau^2} + x_0^3 \right] = 0$$

$O(\epsilon^2)$.

$\Rightarrow$ zeroth order equation : $\dfrac{d^2 x_0}{d\tau^2} + x_0 = 0 \qquad (5)$

First order eqn : $\dfrac{d^2 x_1}{d\tau^2} + x_1 + 2\omega_1 \dfrac{d^2 x_0}{d\tau^2} + x_0^3 = 0 \qquad (6)$

Solve (5) $\Rightarrow x_0(\tau) = A\cos\tau$. (ignore the phase ϕ as we can choose $\tau = 0$ without loss of generality with $\dot{x} = 0$).

Substitute in (6).

$$\frac{d^2x_1}{d\tau^2} + x_1 = +2A\omega_1\cos\tau - A^3\cos^3\tau.$$

Using trigonometric identity for $\cos^3\tau$, we have

$$\underbrace{\frac{d^2x_1}{d\tau^2} + x_1}_{\text{natural freq} = 1} = \left(2A\omega_1 - \frac{3A^3}{4}\right)\underbrace{\cos\tau}_{\text{forcing freq} = 1} - \frac{A^3}{4}\cos 3\tau. \quad (7)$$

$\longleftarrow$ resonance $\longrightarrow$ "Secular term".

BUT, given that we have introduced more unknown coefficients, we can pick those (ω_i) to kill the secular terms.

Set $2A\omega_1 - \frac{3A^3}{4} = 0$

(8)

$$\boxed{\omega_1 = \frac{3A^2}{8}} \Rightarrow$$

$$\omega = 1 + \epsilon\omega_1 + O(\epsilon^2)$$

$$\boxed{\omega = 1 + \frac{3}{8}A^2\epsilon + O(\epsilon^2)}$$

Using 8 in 7, we get

$$\frac{d^2x_1}{d\tau^2} + x_1 = -\frac{A^3}{4}\cos 3\tau .$$

Solving $x_1(\tau) = \frac{A^3}{32}(\cos 3\tau - \cos \tau)$

Thus, we have

$$x(\tau) = x_0(\tau) + \epsilon x_1(t)$$

This is the SOLUTION 9

$$\boxed{x(t) = A\cos\omega t + \frac{\epsilon A^3}{32}(\cos 3\omega t - \cos \omega t)} \boxed{+ O(\epsilon^2)}$$

where $\boxed{\omega = 1 + \frac{3A^2}{8}\epsilon + O(\epsilon^2)}$ 10

This is the AMPLITUDE DEPENDENCE of oscillation frequency

Notice that 9 has period $2\pi/\omega$.

and 10 gives the amplitude dependence of the frequency or period.

– We can continue this procedure to whatever order of ϵ. (use Symbolic toolbox)