Method of averaging

Say we have an ODE

$$\dot{x} = \epsilon f(x, t, \epsilon), \quad x(0) = x_0 \qquad (1)$$

and f is periodic in t. with period T.

Then, we have the averaged equation

$$\dot{z} = \epsilon \bar{f}(z) \qquad z(0) = x_0$$

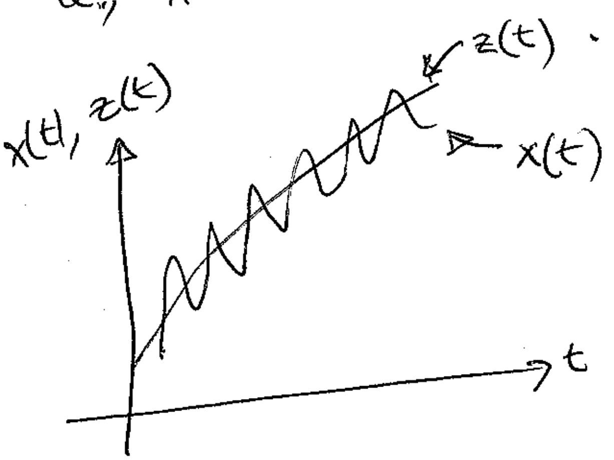
$$\text{where} \quad \bar{f}(x) = \frac{1}{T}\int_0^T f(x, s, 0)\, ds. \qquad (2)$$

— $x(t)$ and $z(t)$ are related as

$$x(t) = z(t) + O(\epsilon), \quad \text{when} \quad t \leq \frac{L}{\epsilon}$$

with L not ϵ dependent.

ie., x & z move together for long, but not too long.



Thus, using 'averaging', we hope to capture the overall "trend" without worrying about the small oscillations about the overall trend.

In many situations, we don't have the equation in the form 1 with time-periodicity but we can get to this form by using "the method of variation of parameters".

Method of averaging applied to the van der Pol oscillator.

(1) $\ddot{x} + x + \epsilon\dot{x}(x^2-1) = 0$ ← not quite in standard form $\dot{y} = \epsilon f(x,t,\epsilon)$ so use method of variation of parameters

When $\epsilon = 0$, the equation is

(2) $\ddot{x} + x = 0$.

whose solution is $x = A\cos(t-\phi)$. (3)

where A and ϕ are constants.

Assume that when $\epsilon \neq 0$ the solution can be represented

(4) by $x(t) = A(t)\cos(t+\phi(t))$ ← this assumption is called "the method of variation of parameters" because it allows "constants A & ϕ to vary"

Note: we've replaced one unknown function of time $x(t)$ with 2 unknown functions of time $A(t)$ and $\phi(t)$. So we are allowed to posit some (arbitrary) relation between $A(t)$ and $\phi(t)$.

$$\dot{x} = \dot{A}\cos(t+\phi) + A\left(-\sin(t+\phi)\right) \quad [1+\dot{\phi}]$$

$$= \dot{A}\cos(t+\phi) - A\dot{\phi}\sin(t+\phi) - A\sin(t+\phi)$$

Let us now arbitrarily choose

$$\boxed{\dot{A}\cos(t+\phi) - A\dot{\phi}\sin(t+\phi) = 0} \quad (5)$$

to be our one free condition

So that $\dot{x}$ becomes

$$\dot{x} = -A\sin(t+\phi) \quad (6)$$

$$\ddot{x} = -\dot{A}\sin(t+\phi) - A\cos(t+\phi) - A\cos(t+\phi)\dot{\phi} \quad (7)$$

Note 1: condition (5) ensures that the expression for $\ddot{x}$ does NOT contain any $\ddot{A}$ or $\ddot{\phi}$ terms. This is just a assumption of convenience. A different condition is Okay, but may be more complicated to deal with, analytically.

Note 2: You will find that many books will directly assume (6) without explaining the rationale, as here.

Now use (4), (6), (7) in (1)

$$\ddot{x} + x + \epsilon \dot{x}(x^2-1) = 0$$

$$\Rightarrow \quad -\dot{A}\sin(t+\phi) - \cancel{A\cos(t+\phi)} - \dot{\phi}A\cos(t+\phi) + \cancel{A\cos(t+\phi)}$$
$$+ \epsilon\left(-A\sin(t+\phi)\right)\left[A^2\cos^2(t+\phi) - 1\right] = 0$$

$$-\dot{A}\sin(t+\phi) - A\dot{\phi}\cos(t+\phi) - A\epsilon\sin(t+\phi)\left[A^2\cos^2(t+\phi) - 1\right] = 0. \quad (8)$$

(5) and (8) are coupled ODEs for $A(t)$ and $\phi(t)$.

Let us solve them first for $\dot{A}$ and $\dot{\phi}$.

(5) sin + (8) cos. gives

$$\left.\begin{aligned} &+ \dot{A}\cancel{\cos(t+\phi)\sin(t+\phi)} - A\dot{\phi}\sin^2(t+\phi) \\ &- \cancel{\dot{A}\sin(t+\phi)\cos(t+\phi)} - A\dot{\phi}\cos^2(t+\phi) \\ &\qquad - A\epsilon\sin(t+\phi)\cos(t+\phi)\left[A^2\cos^2(t+\phi) - 1\right] \end{aligned}\right] = 0.$$

$$+ A\dot{\phi}(1) + A\epsilon\sin(t+\phi)\cos(t+\phi)\left[A^2\cos^2(t+\phi) - 1\right] = 0. \quad (9)$$

Similarly (5) cos(-) $-$ (8) sin (-)

$$\dot{A}\cos^2(t+\phi) - \cancel{A\dot{\phi}\cos(t+\phi)\sin(t+\phi)}$$
$$+ \dot{A}\sin^2(t+\phi) + \cancel{A\dot{\phi}\sin(t+\phi)\cos(t+\phi)}$$
$$+ A\epsilon\sin^2(t+\phi)\left[A^2\cos^2(t+\phi)-1\right] = 0$$

$$\dot{A} + A\epsilon\sin^2(t+\phi)\left[A^2\cos^2(t+\phi)-1\right] = 0. \quad (10)$$

To repeat (9) & (10)

$$(9) \equiv \quad \dot{\phi} = -\epsilon\sin(t+\phi)\cos(t+\phi)\left[A^2\cos^2(t+\phi)-1\right]$$

$$(10) \equiv \quad \dot{A} = -\epsilon A\sin^2(t+\phi)\left[A^2\cos^2(t+\phi)-1\right].$$

Recall the "averaging theorem" which says that

$\dot{z} = \epsilon f(t,z)$ can be replaced with

$$\dot{z} = \epsilon \bar{f}(z) \quad \text{where} \quad \bar{f}(z) = \frac{1}{T}\int_0^T f(t,z).$$

averaged equation.

So we apply this theorem to (9) (10) where

$$Z = \begin{bmatrix} A \\ \phi \end{bmatrix}.$$

The "averaged versions" of A and ϕ will satisfy the following averaged ODEs

$$\dot{\phi} = -\epsilon \frac{1}{2\pi}\int_0^{2\pi} \left[\sin\theta \cos\theta\right] \cdot \left[A^2\cos^2\theta - 1\right] d\theta \qquad (11)$$

$$\dot{A} = -\epsilon \cdot \frac{1}{2\pi} \int_0^{2\pi} A \sin^2\theta \left[A^2\cos^2\theta - 1\right] d\theta. \qquad (12)$$

where $t + \phi = \theta$, $dt = d\theta$.

— We can integrate (11) and (12) either with pencil & paper (integration by parts or trig identities)

or using symbolic MATLAB.

— We get (13) $\boxed{\dot{\phi} = 0}$ ← (can be proved without detailed integration based on odd integrand. Also $\int_0^{2\pi} = \int_{-\pi}^{\pi}$)

$$\boxed{\dot{A} = -\frac{\epsilon}{2\pi} \cdot \frac{\pi A (A^2 - 4)}{4}} \qquad (14)$$

(14) is now an ODE for the "amplitude" $A(t)$ of an oscillation. $x(t) = \underline{A(t)} \cos(t+\phi)$.

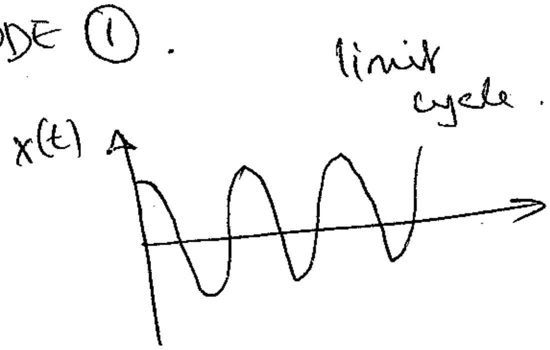
Consider fixed points of the ODE (14).

$$A(A^2-4) = 0.$$

$A=0$ corresponds to the fixed point of ODE (1) namely $x=0$, $\dot{x}=0$.

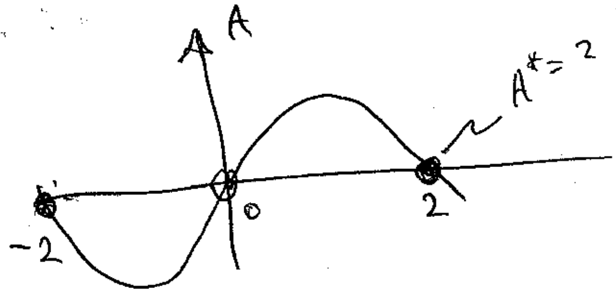
(or) $A^2-4=0$ (or) $A^*=2$ (ignoring -2).

$A^*=2$ is a <u>stable</u> fixed point of (14) and corresponds to a <u>limit cycle</u> (asymptotically stable) of the ODE (1).



$$\boxed{\begin{array}{c}\text{Steady state} \\ x(t) \simeq 2\cos(t+\phi) \\ \phi = \text{constant.}\end{array}}$$

and the ODE (14) gives dynamics when not on the limit cycle.



$A^* = 2$ is a limit cycle.

$\equiv$