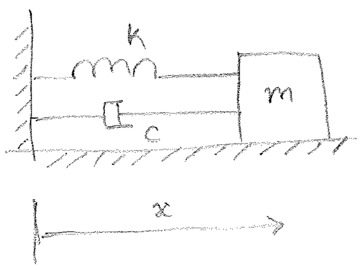
ME832 LECTURE 2

FIRST ORDER ODES IN ONE VARIABLE



LINEAR SPRING MASS DAMPER.

$$m\ddot{x} + c\dot{x} + kx = 0. \quad - \text{(1)}$$

NONLINEAR SPRING + LINEAR DAMPER.

$$m\ddot{x} + c\dot{x} + f(x) = 0 \quad - \text{(2)}$$

Now say that mass m is <u>very small</u> $(m \approx 0)$ and damping c is <u>large</u>. In this limit, Eqn (2) can be <u>approximated</u> by setting the first term to zero.

$$\cancel{m\ddot{x}} + c\dot{x} + f(x) = 0$$

$$c\dot{x} + f(x) = 0$$

$$\dot{x} = -\frac{f(x)}{c}. \quad \text{More generally} \quad \boxed{\dot{x} = h(x)} \quad - \text{(3)}$$

<u>Note 1</u>: What is <u>large damping</u>? Imagine swimming through honey / molasses (for example).

The above approximation is very good in the limit of low Reynold's numbers: "Stokes flow".

Eg. Edward Purcell, Life at low Reynold's numbers.
(classic paper) "ARISTOTELIAN DYNAMICS" (NO INERTIA)

<u>Note 2</u>: APPARENT paradox. Eqn (2) requires 2 initial conditions $x(0), \dot{x}(0)$ and is a second order system.

Eqn (3) is a first order system and needs only 1 initial condition $x(0)$. We will resolve this paradox later, and show in what sense (3) is an approximation of (2).

For now, let us simply study the properties of the differential equations $\dot{x} = h(x)$ where x is a <u>scalar</u>.

<u>Note 3</u>. Note that (3) is an approximation of (2). It is different from reducing (2) to 2 first order ODEs by introducing new variables.

$$\boxed{\dot{x} = h(x)}$$

$\dfrac{dx}{dt} = h(x)$. Initial condition $x(0) = x_0$.

<u>CLOSED FORM SOLUTIONS</u>

by "separating the variables"

$$\int \frac{dx}{h(x)} = \int dt$$

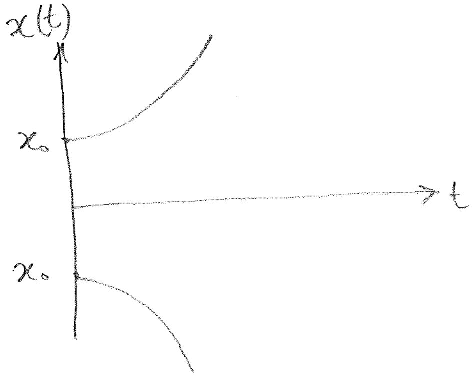
] USEFUL but does not give immediate insight always.

Eg. $\dot{x} = kx \Rightarrow h(x) = kx. \Rightarrow \int_{x_0}^{x} \frac{dx_1}{kx_1} = \int_0^t dt$

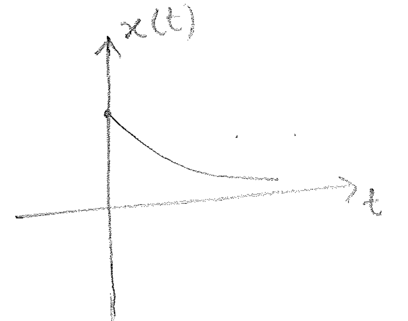
$$\Rightarrow \frac{1}{k} \ln\left(\frac{x}{x_0}\right) = t$$

$$\Rightarrow \boxed{x = x_0 e^{kt}}$$

$k > 0$



$k < 0$



QUALITATIVE / GEOMETRIC ANALYSIS

Say. $\dot{x} = kx$ with $k > 0$.

when $x > 0$, $\dot{x} = kx > 0$ x increases.

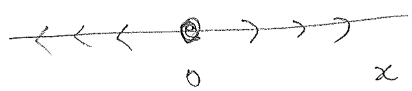
So if $x(0) = x_0 > 0$, x keeps increasing

when $x < 0$, $\dot{x} = kx < 0$ x decreases.

if $x(0) = x_0 < 0$, x keeps decreasing

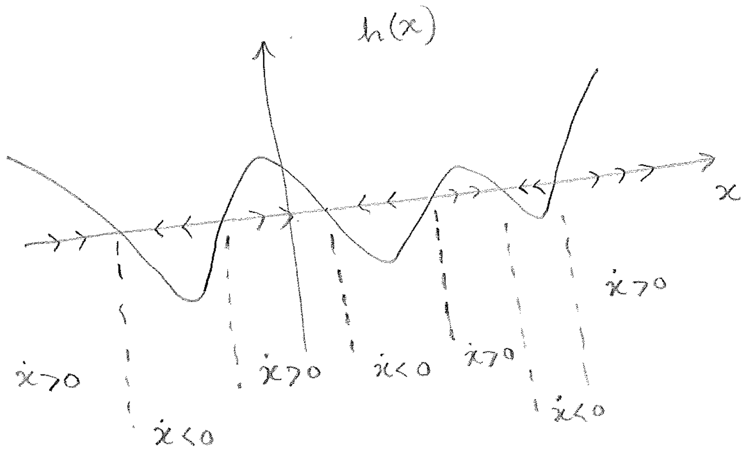
when $x = 0$, $\dot{x} = kx = 0$

if $x(0) = 0$, $x(t) \equiv 0$ $\forall t$.



arrows indicate whether x increases or decreases.

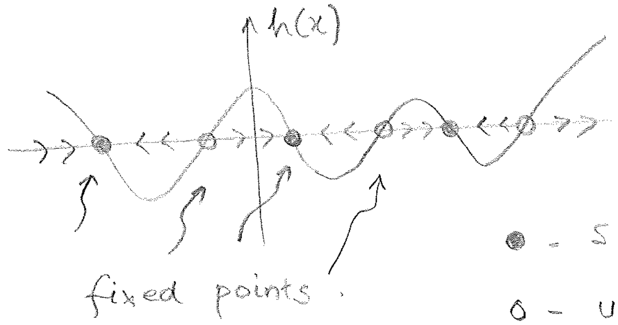
Say. $\dot{x} = h(x)$



FIXED POINTS / EQUILIBRIA

Points x^* at which $\dot{x} = 0$, so that $x(t) = x^*$ for all time if $x(0) = x^*$.

They are found by setting $\dot{x} = h(x^*) = 0$ (where $h(x)$ intersects the x-axis)

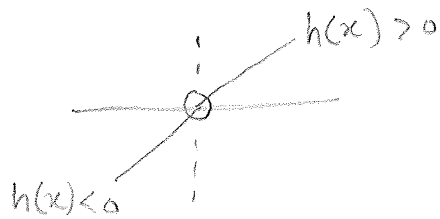


● - STABLE FIXED POINT
o - UNSTABLE FIXED POINT.
} NOTATION.

$x = x^*$ is UNSTABLE FIXED POINT if $h(x^*) = 0$ and in the neighborhood of x^* the dynamics ("flow") is away from x^*.

"LINEAR STABILITY" TEST.

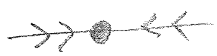
So locally, $h(x)$ should look like



That is, UNSTABLE if

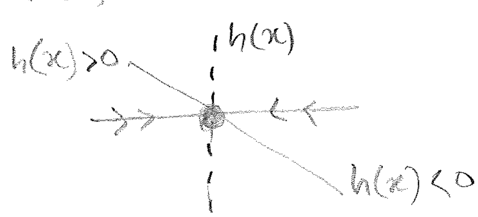
$$h'(x^*) = \frac{dh}{dx}(x^*) > 0$$

$x = x^*$ is STABLE FIXED POINT if $h(x^*) = 0$ and in the neighborhood of x^*, the dynamics is toward x^*.



"LINEAR STABILITY" TEST.

Locally $h(x)$ should look like



That is, STABLE if

$$h'(x) = \frac{dh}{dx}(x^*) < 0.$$

EQUIVALENT TERMS

FIXED POINTS ≡ EQUILIBRIUM ≡ SINGULAR POINTS

LINEARIZATION NEAR A FIXED POINT

$$\dot{x} = h(x). \quad - (A)$$

Say x^* is a fixed point.

So $h(x^*) = 0$.

Expanding $h(x)$ in Taylor series about $x = x^*$,

$$h(x) = \cancelto{0}{h(x^*)} + (x-x^*)\,h'(x^*) + (x-x^*)^2\,h''(x^*) + O\left((x-x^*)^2\right).$$

So to first order $h(x) \approx (x-x^*)\,h'(x^*)$. when $h'(x) \neq 0$

Using this in the original ODE, we have

$$\dot{x} = (x-x^*)\,h'(x^*) \quad - (B)$$

Define $\eta = x - x^*$ - DISTANCE FROM x^*.

$$\dot{\eta} = \dot{x}$$

So $\boxed{\dot{\eta} = h'(x^*)\,\eta}$ - (C) (from (B)) $\leftarrow$ LINERIZATION OF (A)

This is of the form $\dot{\eta} = k\eta$ where $k = h'(x^*)$

STABLE if $k < 0$ i.e., $h'(x^*) < 0$
UNSTABLE if $k > 0$. i.e., $h'(x^*) > 0$

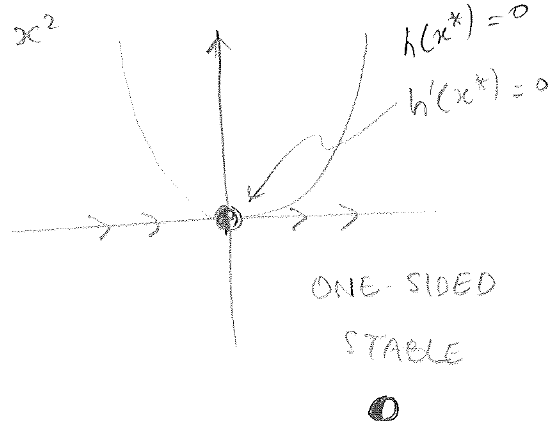
LINEAR STABILITY ANALYSIS

Note 1: Of course, (C) is a good approximation to (A) only when close to x^*.

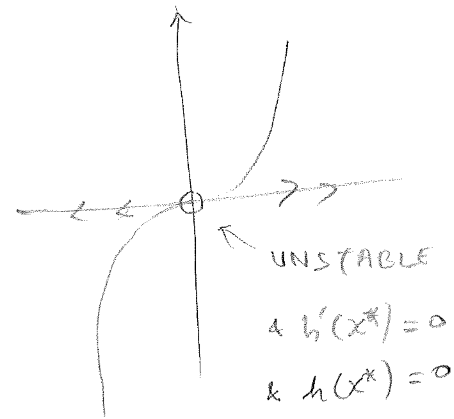
Note 2: Linear stability analysis only says whether small deviations from x^* (namely $\eta = x - x^*$) grow or decay. It does not say whether large deviations will converge to x^*.

<u>Note 3</u> If $h'(x^*) = 0$, there is not enough information in the linearization to decide stability.

e.g. $\dot{x} = x^2$



$\dot{x} = x^3$



Impossibility of oscillations for $\dot{x} = h(x)$, x - scalar

<u>Proof:</u> An oscillation requires x to increase and decrease. That is, at some x, $\dot{x}$ should be first positive, then negative.

But this is impossible because at any given x $\dot{x} = h(x)$ is fixed and its sign is uniquely determined.

$\Rightarrow$ NO OSCILLATIONS.

The absence of oscillations agrees with how we derived $\dot{x} = h(x)$ as an "OVERDAMPED" system with no inertia.

The ONLY BEHAVIOR POSSIBLE is monotonic increase (or) decrease (or) fixed points!

BUCKLED BEAM

Example

Consider the 2^{nd} order ODE

$$m\ddot{x} + c\dot{x} \underbrace{-\mu x + \lambda x + x^3}_{\text{NONLINEAR SPRING-LIKE TERM.}} = 0 \quad - (1)$$

ASIDE

This ODE is an approximation of the transverse vibration dynamics of a column buckling

See for instance
"Stability problems in applied mechanics"
by A.K. Mallik & J.K. Bhattacharjee (2005)

The ODE (1) is obtained by first writing a nonlinear PDE and then "projecting" onto the "first" linear mode.

Say $m \approx 0$, so that we have the simpler first order scalar ODE

$$c\dot{x} - \mu x + \lambda x + x^3 = 0 \quad - (2)$$

For simplicity, set $c = 1$.

$$\text{So } \dot{x} = (\mu - \lambda)x - x^3 \quad - (3)$$
$$= h(x) \text{ say}$$

FIXED POINTS

Set RHS of (3) to zero.

$$(\mu - \lambda)x - x^3 = 0$$

$$x\left[\mu - \lambda - x^2\right] = 0$$

So Fixed points are $x^* = 0$, $x^* = \pm\sqrt{\mu - \lambda}$ ← these solutions are real only when $\mu - \lambda > 0$

So when $\mu - \lambda < 0$, 1 FIXED POINT, $x^* = 0$

when $\mu - \lambda > 0$, 3 FIXED POINTS, $x^* = 0$, $x^* = \pm\sqrt{\mu - \lambda}$

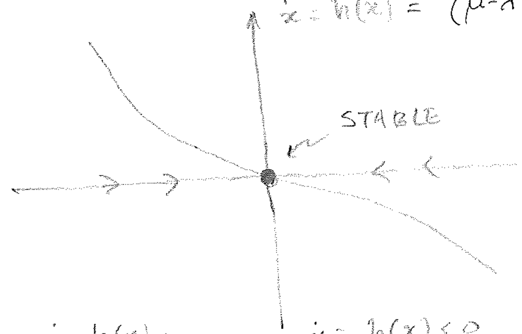
STABILITY USING PICTURES

Case 1

$\mu - \lambda < 0$

(For simplicity consider, $\mu - \lambda = -1$)

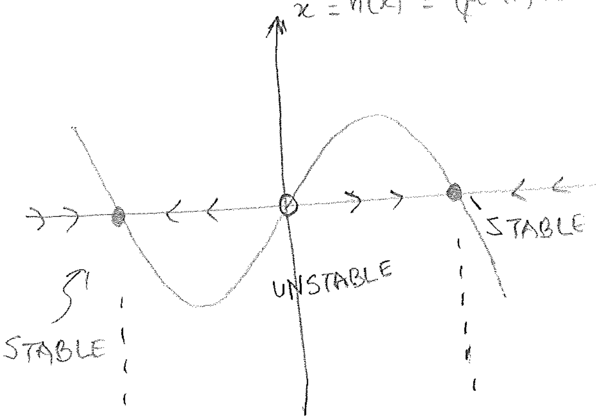
$\dot{x} = h(x) = (\mu-\lambda)x - x^3$ ← negative $\forall\, x > 0$, positive $\forall\, x < 0$ } and magnitude increases with x



Case 2

$\mu - \lambda > 0$

$\dot{x} = h(x) = (\mu-\lambda)x - x^3$

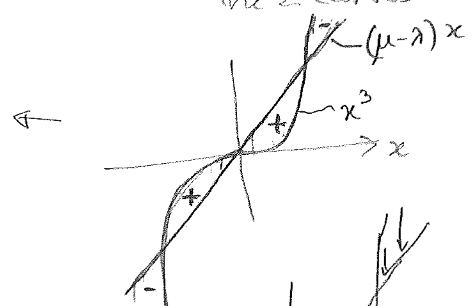


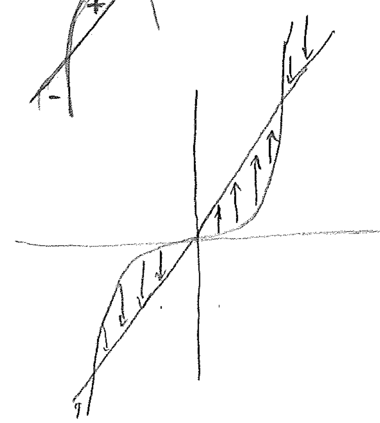
How to plot $(\mu-\lambda)x - x^3$?

→ Plot $(\mu-\lambda)x$

→ Plot x^3

→ $(\mu-\lambda)x - x^3$ is the difference between the 2 curves





COMPUTING STABILITY BY LINEARIZATION

$$\frac{dh}{dx} = \mu - \lambda - 2x^2.$$

Case 1 $\mu - \lambda < 0$.

$\boxed{x^* = 0}$ $\quad \frac{dh}{dx}(x^* = 0) = \mu - \lambda < 0 \Rightarrow$ STABLE

<u>case 2</u> $\mu - \lambda > 0$

$\boxed{x^* = 0}$ $\quad \frac{dh}{dx}(x^* = 0) = \mu - \lambda > 0 \Rightarrow$ UNSTABLE

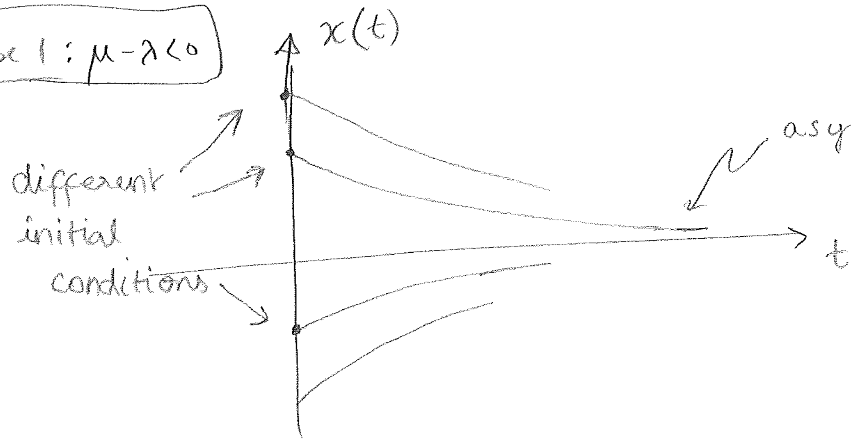
$\boxed{x^* = \pm\sqrt{\mu - \lambda}}$ $\quad \frac{dh}{dx} = \mu - \lambda - 2x^2$

$$= \mu - \lambda - 2(\mu - \lambda)$$

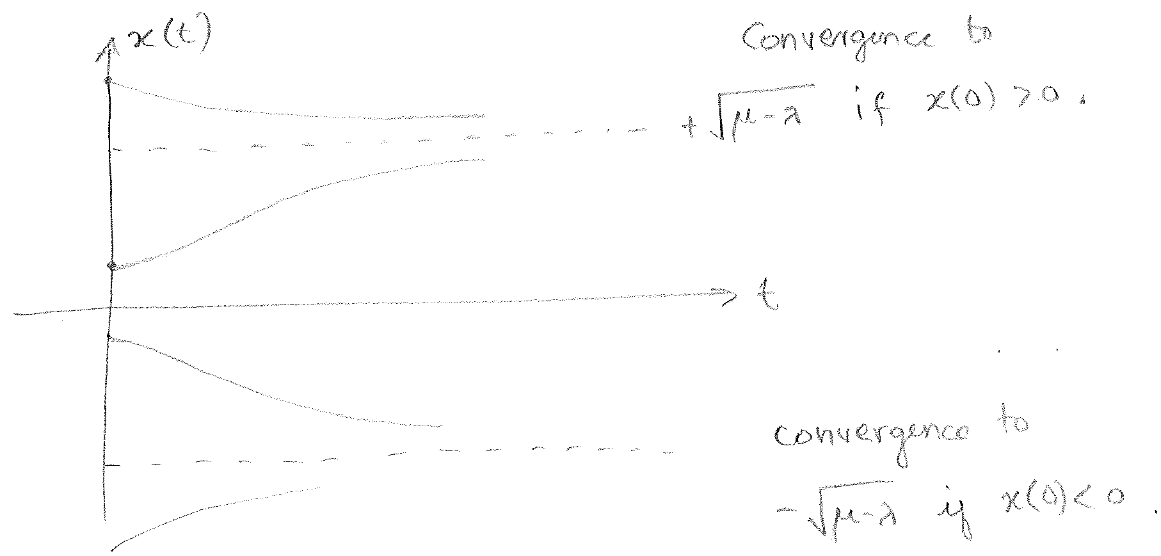
$$= -(\mu - \lambda) < 0 \Rightarrow \text{STABLE}$$

WHAT DO SOLUTIONS $x(t)$ LOOK LIKE?

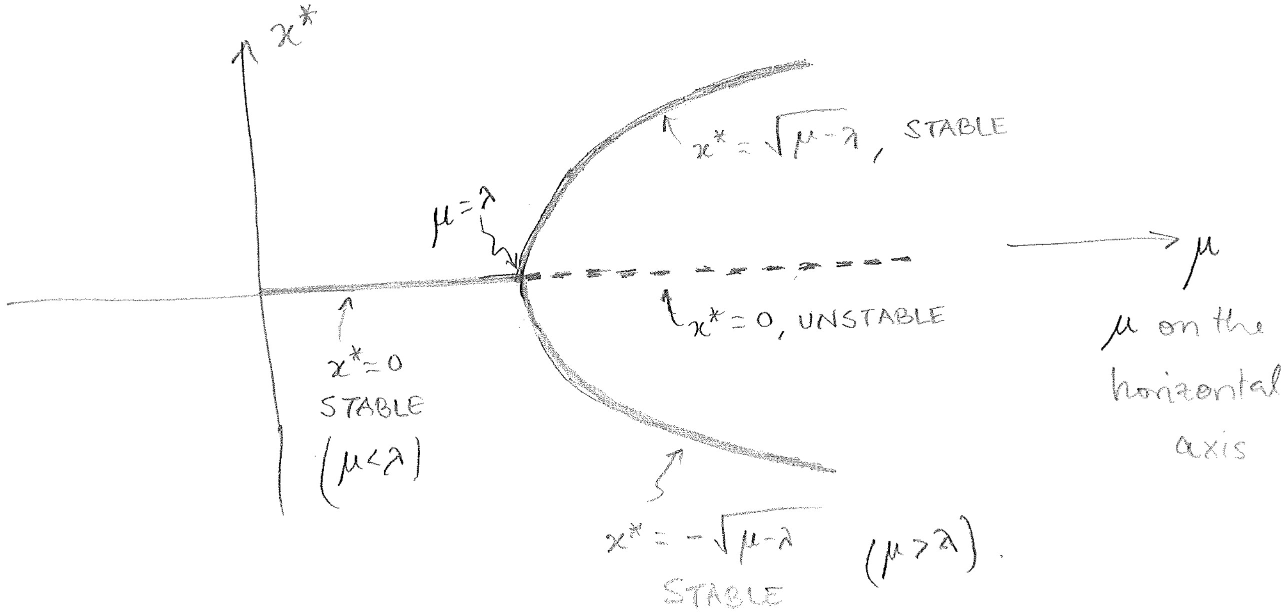
$\boxed{\text{case 1: } \mu - \lambda < 0}$



<u>case 2</u>



PLOTTING ALL FIXED POINTS x^* AS A FUNCTION OF μ

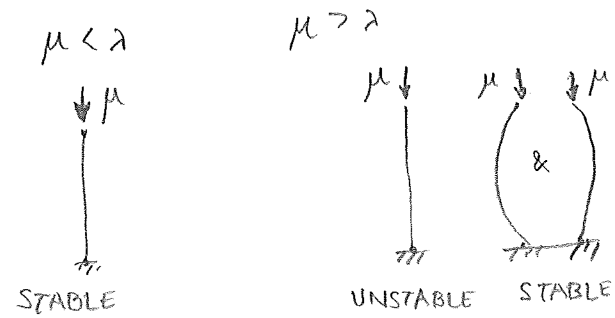


Such a plot is called a

<u>BIFURCATION DIAGRAM</u>.

The change in qualitative behavior as you change μ across λ is called a <u>bifurcation</u>. $\mu = \lambda$ is a <u>bifurcation point</u>.

[Going back to the <u>buckling analogy</u>, μ is like the "vertical" force on the column, $x^* = 0$ is the unbuckled state which is stable for $\mu < \lambda$, $x^* = \pm\sqrt{\mu - \lambda}$ are the buckled equilibria which exist only for $\mu \geq \lambda$.]



Bifurcations (Intro/ Rough outline).

Colloquially, a bifurcation is a qualitative change in the dynamics of a system when some system parameter is changed. More formally, a bifurcation is said to have occurred when the phase portrait's "topological structure" changes when a parameter is varied.

Example: change in the number and stability of fixed points.

Bifurcations in 1D

$\dot{x} = f(x, r)$ ← parameter being changed.

- Saddle-node bifurcation

As the parameter r is changed, an unstable fixed point and a stable fixed point come together and "vanish".

Eg.. $\dot{x} = r + x^2$ ← canonical example.

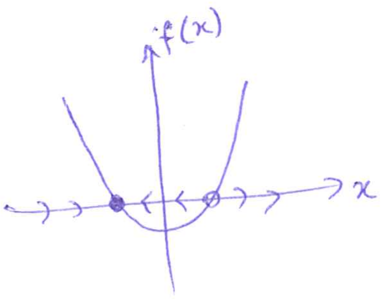
~~when~~ Fixed points : $r + x^2 = 0$

$x^2 = -r$.

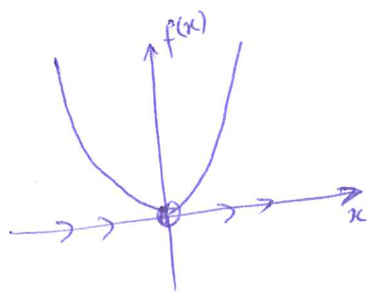
when $r < 0$, 2 FPs. $x^* = \pm\sqrt{|r|}$.

when $r = 1$, 1 FP $x^* = 0$

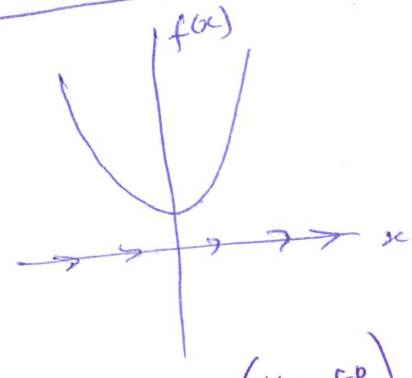
when $r > 0$, 0 FPs.



$r < 0$ (2 FPs)

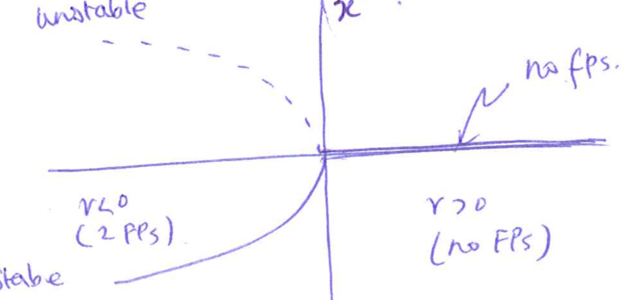


$r = 0$ (1 FP)



$r > 0$ (No FP).

Bifurcation diagram.



Called a "saddle-node" because in 2D, the unstable FP is a saddle and the stable FP is a node.

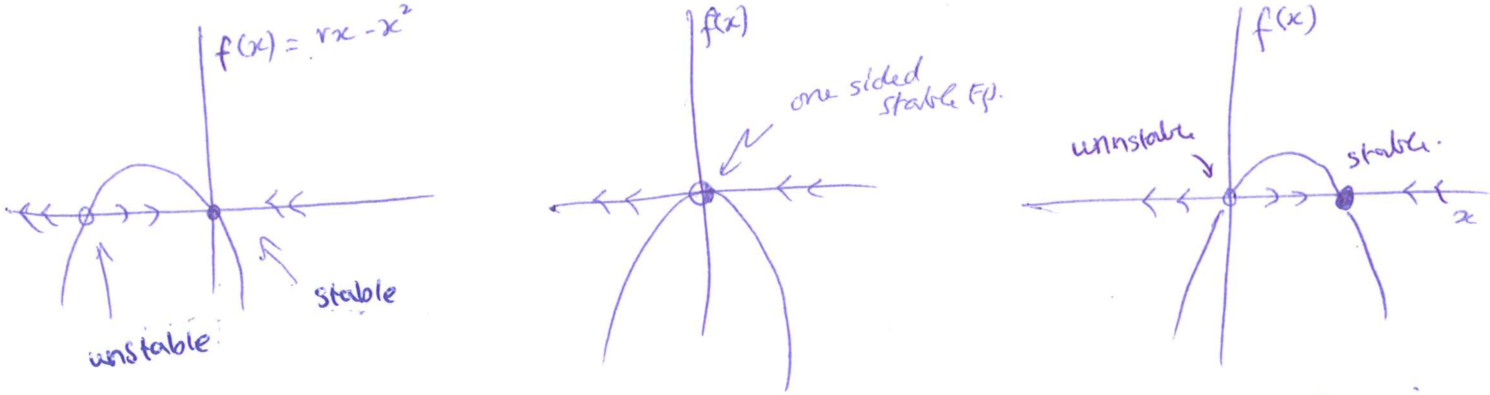
also called <u>fold - bifurcation</u> (or) <u>turning point bifurcation</u>

TRANSCRITICAL BIFURCATION

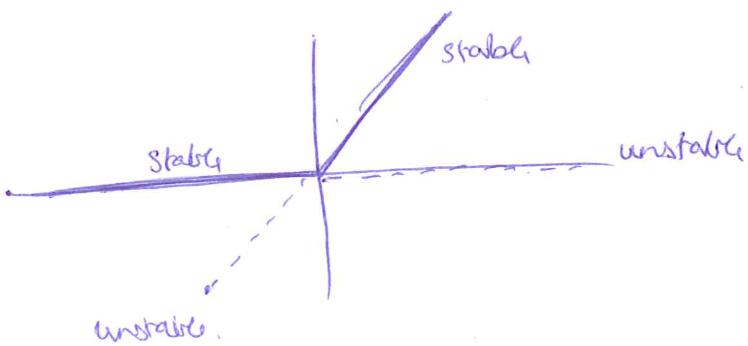
Eg. $\dot{x} = rx - x^2 = x(r-x)$. ← canonical example.

Fixed points : $x^* = 0$ and $x^* = r$.

when $r < 0$ ($x^* = 0$ & $x^* < 0$.) when $r = 0$ (both FPs coincide)



Bifurcation diagram.



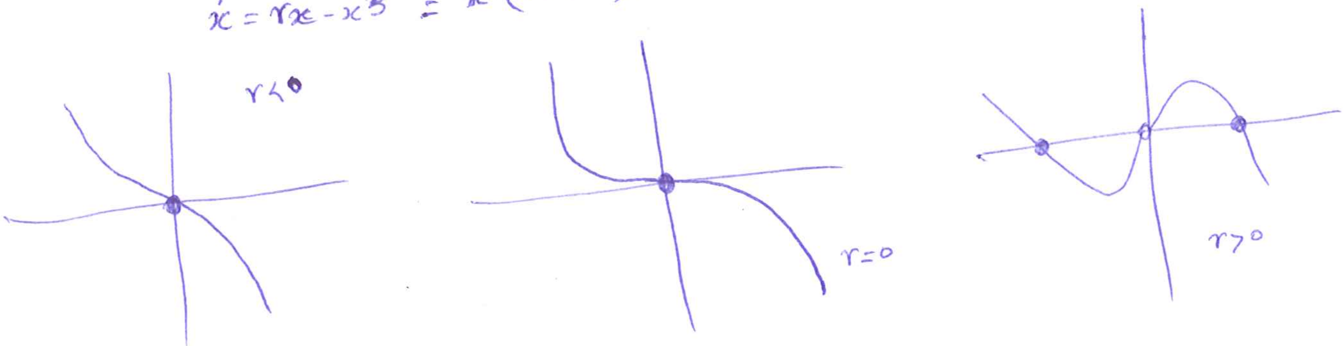
as the parameter r is varied, two fixed points, collide and 'exchange stability

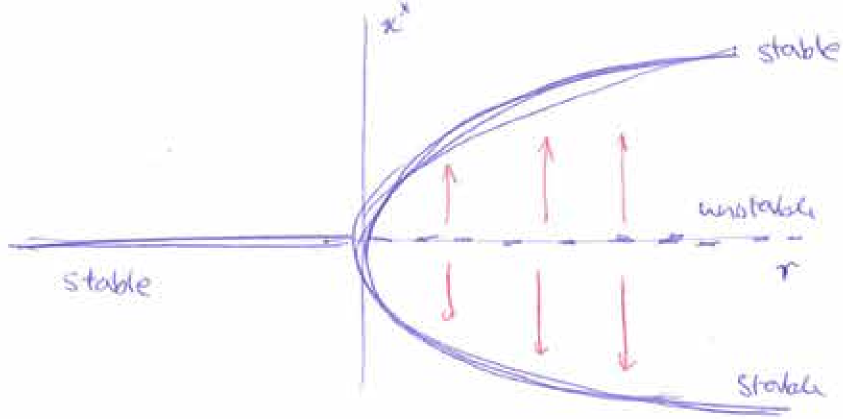
PITCHFORK BIFURCATION

Supercritical Pitchfork :

$\dot{x} = rx - x^3 = x(r - x^2)$

(FPs : $x^* = 0$ if $r < 0$
& $x^* = 0$, $x^* = \pm\sqrt{r}$ if $x > 0$

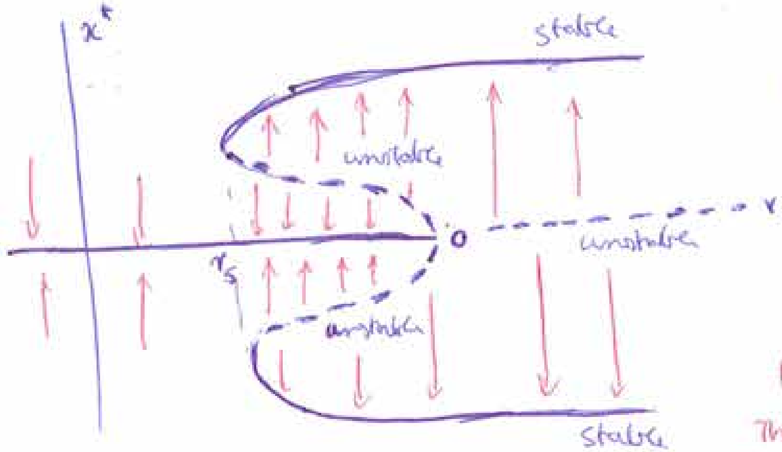




as r is changed across 0, $x^*=0$ goes unstable. the system is at $x^*=0$ for $r<0$ and then tracks $x^* = \pm\sqrt{r}$ as r is changed to greater than zero. The change in x^* is gradual and continuous

Subcritical pitchfork bifurcation

eg: $\dot{x} = rx + x^3 - x^5$



$r < r_s$. 1 stable FP

$r_s < r < 0$. 5 FPs. 2 unstable 3 stable

$0 < r < \infty$ 3 FPs 2 stable 1 unstable ($x^*=0$)

Say $r<0$ and your system is at $x^*=0$.
Now r is increased to $r>0$.
The system will now go to one of the new stable fixed points. These fixed points are a finite distance away from $x^*=0$ – and this can be catastrophic sometimes.