General 2^{nd} order ODE (LINEAR)

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

FIXED POINT : $x = 0$.

Hypothesis for solution. $x = ve^{\lambda t}$ $\quad \dot{x} = v\lambda e^{\lambda t}$ $\quad \ddot{x} = v\lambda^2 e^{\lambda t}$.

$$\Rightarrow \lambda^2 \cancel{v e^{\lambda t}} + \frac{c}{m}\lambda \cancel{v e^{\lambda t}} + \frac{k}{m}\cancel{v e^{\lambda t}} = 0$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

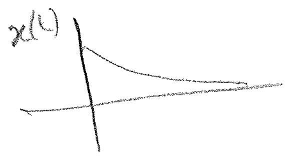
Two solutions] $$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

General solution for $x(t)$:

$$x(t) = v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t}.$$

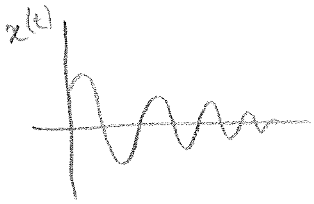
when positive damping and stiffness
when $c > 0$ and $k > 0$, $x = 0$ is STABLE. (x decays to $x = 0$)

when $c^2 - 4km > 0$, $\sqrt{c^2 - 4km}$ is real and



the solution consists of exponentials and no oscillations. (OVERDAMPED)

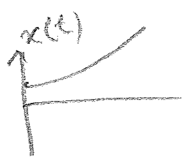
when $c^2 - 4km < 0$, $\sqrt{c^2 - 4mk}$ is complex and



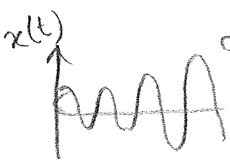
the solution consists of sines, cosines and an exponential.

$\Rightarrow$ Oscillatory solutions (UNDER DAMPED)

when $c < 0$ or $k < 0$ atleast one of λ_1 and λ_2 has a positive real part. (negative damping or negative stiffness.



$\Rightarrow$ $x = 0$ UNSTABLE.



again, when $c^2 - 4mk > 0$, no oscillations + growth

when $c^2 - 4mk < 0$, oscillatory growth

LINEAR SYSTEMS IN 2D

The stability or otherwise of a fixed point / equilibrium of a nonlinear system is "usually" (in a sense that will be made precise) determined by analysing the linearization.

So we first study linear systems in detail.

General 2D linear system:

$$\frac{dy_1}{dt} = a_{11} y_1 + a_{12} y_2$$

$$\frac{dy_2}{dt} = a_{21} y_1 + a_{22} y_2$$

$$\dot{Y} = AY \quad - (1)$$

General solution for $Y(t)$

Hypothesis $Y = V e^{\lambda t}$ $-(2)$ where $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(2) in (1) gives

$$\lambda V \cancel{e^{\lambda t}} = A V \cancel{e^{\lambda t}}$$

$$\boxed{AV = \lambda V} \quad \text{"Eigenvalue problem"}$$

TRUE for N-DIMENSIONAL SYSTEMS

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$(A - \lambda I) V = 0$

V - Eigenvector

λ - Eigenvalue.

Finding eigenvalues

Eigenvalues satisfy

$$\text{determinant} (A - \lambda I) = 0$$

For 2D,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} = 0$$

$$\Rightarrow \lambda^2 - \lambda (a_{11} + a_{22}) + (a_{11} a_{22} - a_{12} a_{21}) = 0$$

There are 2 solutions λ_1 and λ_2.

$$\lambda^2 - \lambda(\text{trace}(A)) + \det(A) = 0$$

$$\lambda_{1,2} = \frac{+\text{trace}(A) \pm \sqrt{[\text{trace}(A)]^2 - 4\det(A)}}{2}$$

with corresponding eigenvectors V_1 and V_2

Note: We showed that $\dot{Y} = AY$ is equivalent to

$$\ddot{z} + (a_{11} + a_{22})\dot{z} + (a_{11}a_{22} - a_{12}a_{21})z = 0$$

$$\ddot{z} - (\text{trace}(A))\dot{z} + (\det(A))z = 0$$

$-\text{trace}(A) \sim$ damping / mass (can be negative)

determinant $(A) \sim$ stiffness / mass (can be negative)

So general solution is:

$Y(t) =$ a linear combination of $V_1 e^{\lambda_1 t}$ and $V_2 e^{\lambda_2 t}$.

$$= k_1 V_1 e^{\lambda_1 t} + k_2 V_2 e^{\lambda_2 t}$$

k_1 and k_2 are constants that can be determined given initial conditions $Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$.

Let $\tau = \text{trace}(A)$

$\Delta = \text{determinant}(A)$.

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta} \right).$$

<u>case 1</u> : $\tau^2 - 4\Delta > 0$.

$\Rightarrow \lambda_{1,2}$ REAL.

Solution looks like $Y(t) = k_1 V_1 e^{\lambda_1 t} + k_2 V_2 e^{\lambda_2 t}$.

(STABLE). $Y(t)$ decays to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in general only when

both λ_1 and λ_2 are negative.

$$\Rightarrow \quad \tau + \sqrt{\tau^2 - 4\Delta} < 0$$
$$\text{and } \tau - \sqrt{\tau^2 - 4\Delta} < 0$$

$\tau - \sqrt{\tau^2 - 4\Delta} < 0 \Rightarrow \boxed{\tau < 0.}$

and $\tau + \sqrt{\tau^2 - 4\Delta} < 0$ & $\tau < 0 \Rightarrow \boxed{\Delta > 0}$

(UNSTABLE) if either $\tau > 0$ or $\Delta > 0$

In all these cases, because $Y(t)$ consists only of exponentials and no sines and cosines, we have no oscillations.

Case 2 $\quad \tau^2 - 4\Delta < 0$

$\Rightarrow \lambda_{1,2}$ COMPLEX CONJUGATES.

Solution looks like

$$Y(t) = \underbrace{e^{\tau/2\, t}}_{\substack{\text{exponential} \\ \text{(monotonic)}}} \underbrace{\left[k_1 V_1 \cos\sqrt{4\Delta - \tau^2}\, t + k_2 V_2 \sin\sqrt{4\Delta - \tau^2}\, t \right]}_{\text{oscillatory.}}$$

(STABLE) $\quad Y(t)$ decays if $\tau < 0$ and $e^{\frac{\tau t}{2}} \to 0$ as $t \to \infty$

This decay will be oscillatory.

(UNSTABLE) $\quad Y(t)$ grows unbounded if $\tau > 0$ and $e^{\frac{\tau t}{2}} \to \infty$ as $t \to \infty$

This growth will be oscillatory.

LECTURE 4B / LECTURE 5

$$\frac{dY}{dt} = AY \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \qquad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

$\tau = \text{trace}(A) = a_{11} + a_{22}$

$\Delta = \text{determinant}(A) = a_{11}a_{22} - a_{12}a_{21}$.

STABLE

if and only if $\tau < 0$ and $\Delta > 0$.

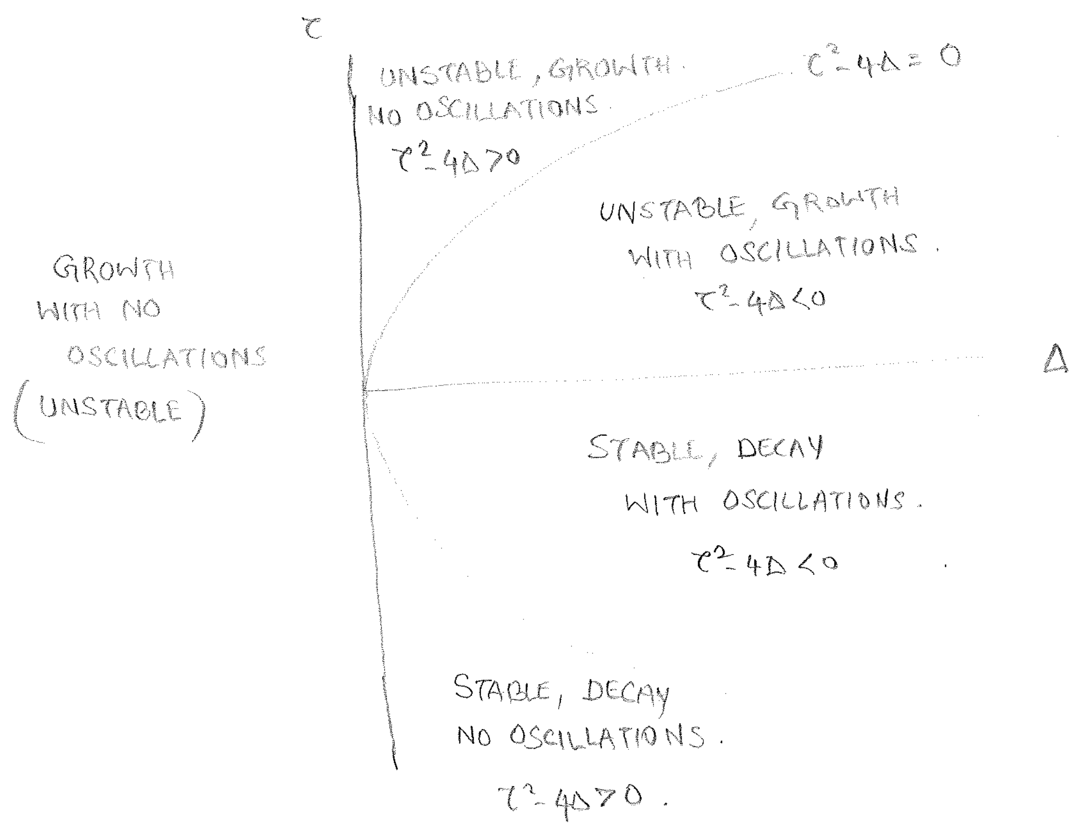
UNSTABLE

if $\tau > 0$ or $\Delta < 0$

OSCILLATIONS if $\tau^2 - 4\Delta < 0$ (like underdamped)

NO OSCILLATIONS if $\tau^2 - 4\Delta > 0$ (like overdamped).

SUMMARY not counting boundary cons.



PHASE PORTRAITS, CLASSIFICATION OF FIXED POINTS, etc.

$\dot{Y} = AY$ or $\dot{Y} = f(Y)$.

$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ - is the <u>state variable</u>. It is the set of all variables whose values need to be specified in general so as to be able to determine the future of system for all time.

<u>State space</u> : set of all values for the state variables.

eg. if $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is the state variable, the state space is $\mathbb{R}^2$, the plane.

In 2D, the <u>state space</u> is also called the <u>phase plane</u>

In ND, the phase space is also called the <u>phase space</u>

In 2D, the qualitative dynamical behavior of the system can be represented graphically by plotting $y_1(t)$ and $y_2(t)$ [in the phase plane] for a number of initial conditions. Such a figure is called a <u>phase portrait</u>. The individual trajectories $y_1(t)$ vs $y_2(t)$ are called <u>phase trajectories</u>.

In ND, one can imagine similar trajectories through the N-dimensional state space.

Example 1

$\dot{Y} = AY$, $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ λ_1 and λ_2 real.

Eigenvalues λ_1 and λ_2.
Eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

General solution $Y(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$.

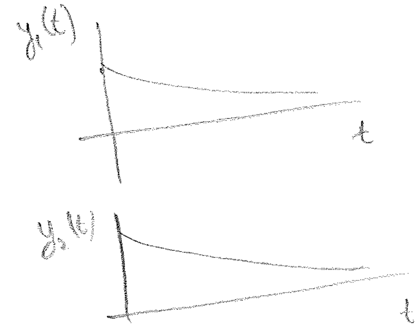
$$\left.\begin{aligned} \dot{y}_1(t) &= \lambda_1 y_1 \\ \dot{y}_2(t) &= \lambda_2 y_2 \end{aligned}\right] \Rightarrow \begin{aligned} y_1 &= y_1(0) e^{\lambda_1 t} \\ y_2 &= y_2(0) e^{\lambda_2 t} \end{aligned}$$

A phase portrait is obtained by plotting $y_1(t)$ vs $y_2(t)$ for various initial conditions.

case 1 $\lambda_1 = \lambda_2 < 0$

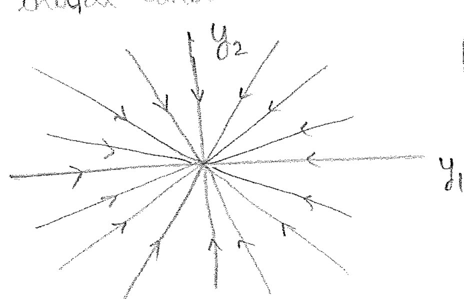
$$y_1(t) = y_1(0) e^{\lambda_1 t} \qquad \frac{y_1(t)}{y_2(t)} = \frac{y_1(0)}{y_2(0)}$$
$$y_2(t) = y_2(0) e^{\lambda_2 t}$$

$$\Rightarrow y_2(t) = \frac{y_2(0)}{y_1(0)} y_1(t)$$



That is, y_1 vs y_2 is a straight line with slope $\frac{y_2(0)}{y_1(0)}$.

The slope depends on initial condition and the line passes through the origin.



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ called a "STABLE NODE"

In particular a "STABLE STAR NODE"

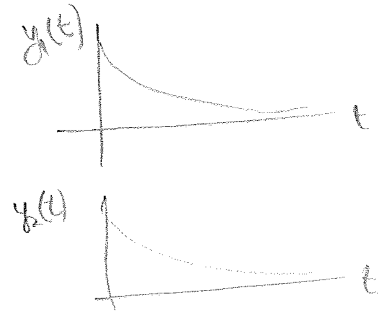
The direction of the arrows indicate what happens to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ as time moves forward. Here, wherever you start (whatever the initial condition), $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

Case 2

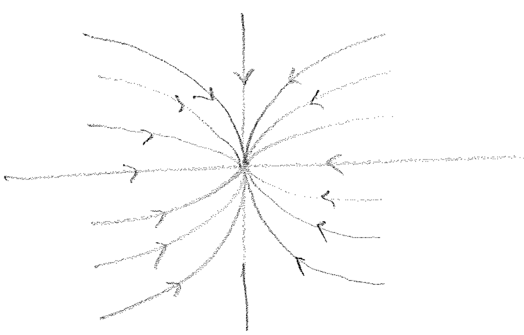
$\lambda_1 < \lambda_2 < 0$

$y_1(t) = y_1(0)\, e^{\lambda_1 t}$.

$y_2(t) = y_2(0)\, e^{\lambda_2 t}$.



$\lambda_1 < \lambda_2 \Rightarrow y_1$ decays faster than y_2, which is reflected in the phase portrait below:

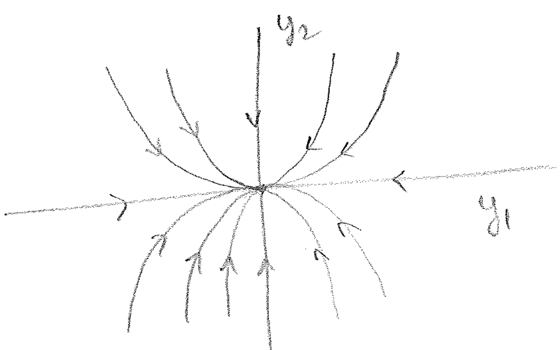


You can also find an explicit relation between y_1 and y_2 by eliminating time t in the general solutions.

The fixed point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is called a STABLE NODE.
(not a STAR though)

Case 3

$\lambda_2 < \lambda_1 < 0$

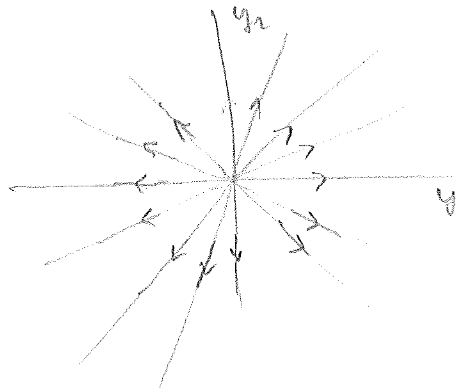


again called a STABLE NODE.

Here $\lambda_2 < \lambda_1 < 0 \Rightarrow y_2$ decays faster than y_1.

<u>Case 4</u>

$\lambda_1 = \lambda_2 > 0$

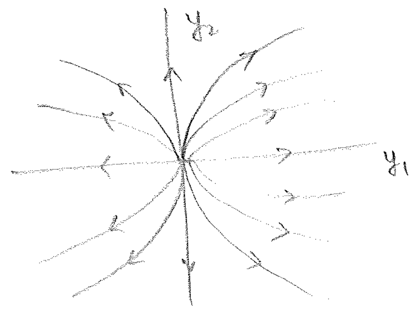


UNSTABLE STAR NODE

y_1 and y_2 grow at the same rate near the fixed point.

<u>Case 5</u>

$\lambda_1 > \lambda_2 > 0$

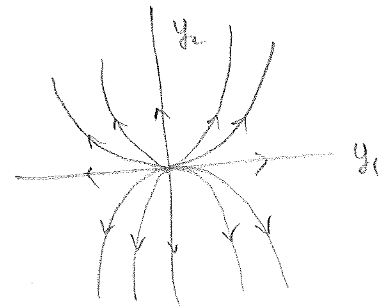


UNSTABLE NODE

y_1 grows faster than y_2.

<u>Case 6</u>

$\lambda_2 > \lambda_1 > 0$



UNSTABLE NODE

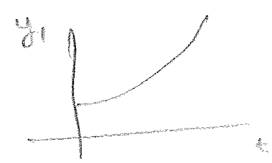
y_2 grows faster than y_1.

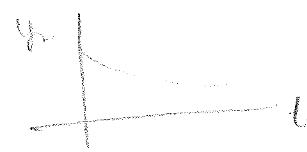
<u>Case 7</u>

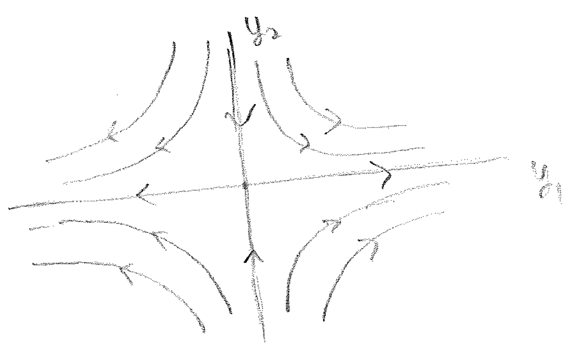
$\lambda_1 > 0$ and $\lambda_2 < 0$

$$y_1(t) = y_1(0)\, e^{\lambda_1 t} \to \infty$$

$$y_2(t) = y_2(0)\, e^{\lambda_2 t} \to 0$$







The FP is called a SADDLE POINT.

The phase portrait is consistent with the following observations.

If you start on the y_1-axis. i.e., $\begin{bmatrix} y_1(0) \\ 0 \end{bmatrix}$, $y_1(t) \to \infty$ and $y_2 = 0$ if $y_1(0) > 0$

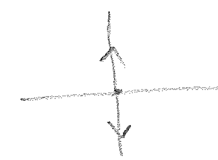
$y_1(t) \to -\infty$ and $y_2 = 0$ if $y_1(0) < 0$

Similarly, if you start on the y_2-axis i.e., $\begin{bmatrix} 0 \\ y_2(0) \end{bmatrix}$, $y_2(t) \to 0$.

Here, the y_1-axis is called the UNSTABLE MANIFOLD. 

More generally, the eigenvector corresponding to the real eigenvalue that is positive gives the UNSTABLE MANIFOLD.

Here it is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Here, the y_2-axis is called the STABLE MANIFOLD. 

For a more general saddle point in 2D, the STABLE MANIFOLD is given by the eigenvector corresponding to the negative real eigenvalue.

Here, given by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Remarks :

1) In nonlinear systems, the stable manifold of a fixed point is defined as a surface (or curve) such that if the initial condition is on the surface, the state asymptotically reaches the fixed point as $t \to \infty$, that is, when time is "run forward".

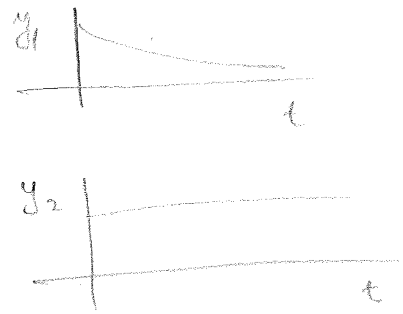
The unstable manifold of a fixed point is defined as a surface (or a curve) such that if the initial condition is on this surface, the state asymptotically reaches the fixed point when time is "run backward" (as $t \to -\infty$).

2) Time running forward is following the arrows in the phase portrait.

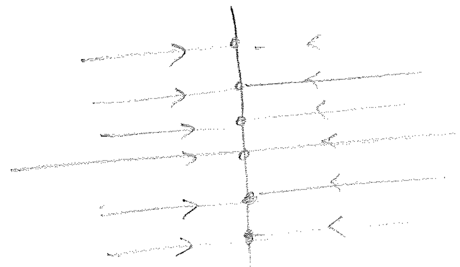
Time running backward is following the reversed arrows, equivalent to hitting the REWIND BUTTON on your DVD player / VCR.

Case 8

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} \dot{y}_1 &= \lambda_1 y_1 \\ \dot{y}_2 &= 0 \end{aligned}$$



Here, the fixed point is not unique.
any point $(0, y_2)$ is a fixed point.



Line of fixed points.

SPECIAL BOUNDARY CASE' Example 2

WHEN THERE ARE NOT ENOUGH EIGENVECTORS

$\dot{Y} = AY$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Usually, we can find eigenvalues λ_1, λ_2 and distinct eigenvectors V_1 and V_2 associated with the eigenvalues.

But sometimes there is only <u>one distinct eigenvector</u>. An example follows.

$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$. Eigenvalues $\lambda^2 - (a+a)\lambda + a^2 = 0$.

$$\lambda_1 = a \ . \ \lambda_2 = a.$$

Eigenvectors V. $\quad AV = \lambda V$.

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} av_1 + v_2 \\ av_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix} \Rightarrow v_2 = 0. \quad \text{and } v_1 \neq 0$$

Eigevector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

This is the <u>only eigenvector</u>.

If we used our <u>usual</u> formula for the general solution for $\dot{Y} = AY$, we'd get the following:

$$Y(t) = c_1 V_1 e^{at}.$$

Because this solution has only one free parameter c_1, it cannot be a general solution.

How to find the general solution?

$$\dot{Y} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} Y \qquad \begin{aligned} \dot{y}_1 &= a y_1 + y_2 \\ \dot{y}_2 &= a y_2 . \end{aligned}$$

$$\Rightarrow \quad y_2 = y_2(0)\, e^{at},$$

$$\dot{y}_1 = a y_1 + y_2(0) e^{at}.$$

$$\dot{y}_1 - a y_1 = y_2(0) e^{at}$$

Integrating factor $= e^{-at}$. Multiplying the above eqn by e^{-at}:

$$e^{-at} \dot{y}_1 - a e^{-at} y_1 = y_2(0)$$

$$\frac{d}{dt}\left(e^{-at} y_1\right) = y_2(0). \Rightarrow \quad e^{-at} y_1 = y_2(0) t + c_1$$

$$y_1(t) = e^{at}\left[y_2(0) t + c_1 \right].$$

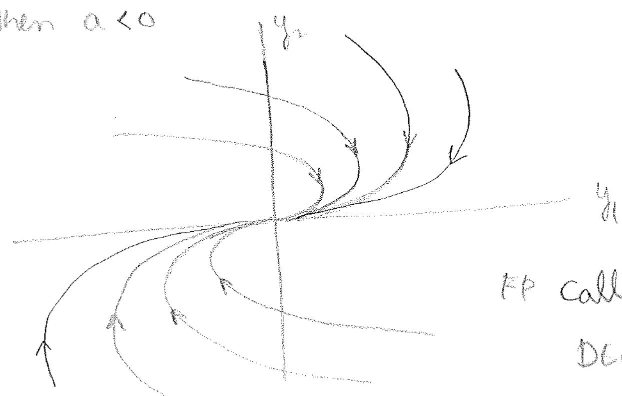
Clearly $c_1 = y_1(0)$. $\Rightarrow$

General solution $\longrightarrow$
$$\boxed{\begin{aligned} y_1(t) &= e^{at}\left[y_1(0) + y_2(0) t \right] \\ y_2(t) &= y_2(0)\, e^{at} \end{aligned}}$$

Phase portrait $\quad \dfrac{y_2(t)}{y_1(t)} = \dfrac{y_2(0)}{y_1(0) + y_2(0) t}$.

Plot y_1 vs y_2 for different initial conditions to get the following.

When $a < 0$



FP called a DEGENERATE NODE

When $a > 0$

what does the phase portrait look like.

(unstable version of the portrait to the left).

Example 3

$$\ddot{x} + c\dot{x} + kx = 0 \qquad c > 0 \text{ and } k > 0$$

$$Y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \dot{Y} = AY$$

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -ky_1 - cy_2 \qquad A = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}.$$

Eigenvalues $\lambda^2 - (-c)\lambda + k\lambda = 0$

$$\lambda^2 + c\lambda + k = 0$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4k}}{2}$$

Say $c^2 - 4k < 0$ so that the eigenvalues are complex conjugates.

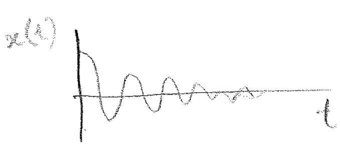
$$\lambda_{1,2} = \frac{-c}{2} \pm i \frac{\sqrt{4k - c^2}}{2}.$$

general solution $x(t) = e^{-\frac{c}{2}t}\left[b_1 \cos \omega t + b_2 \sin \omega t \right]$

where $\omega = \dfrac{\sqrt{4k - c^2}}{2}$.

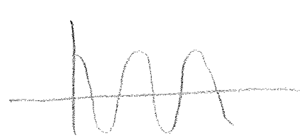
and b_1, b_2 depend on initial conditions.

$c > 0$



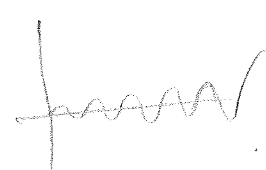
decaying oscillation

$c = 0$

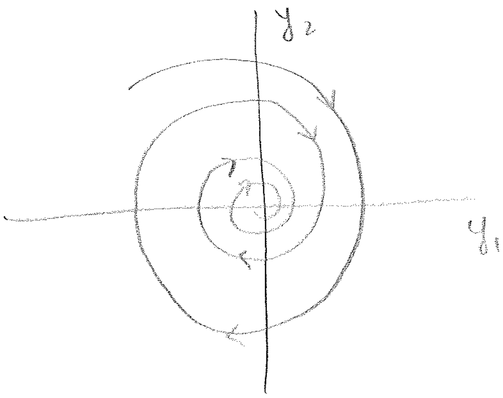


periodic oscillation

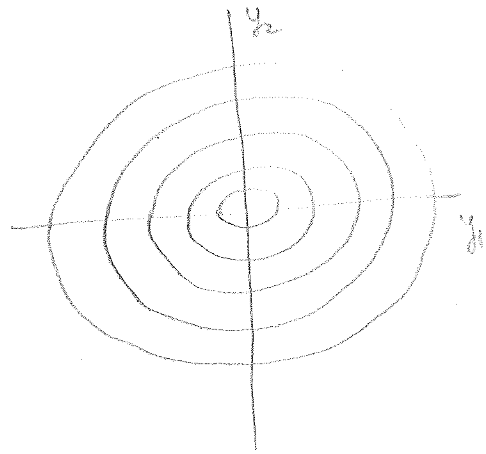
$c < 0$



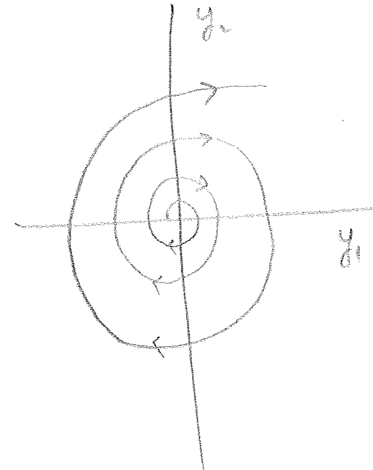
growing oscillation



$\begin{bmatrix}0\\0\end{bmatrix}$ is a <u>STABLE FOCUS / SPIRAL</u>

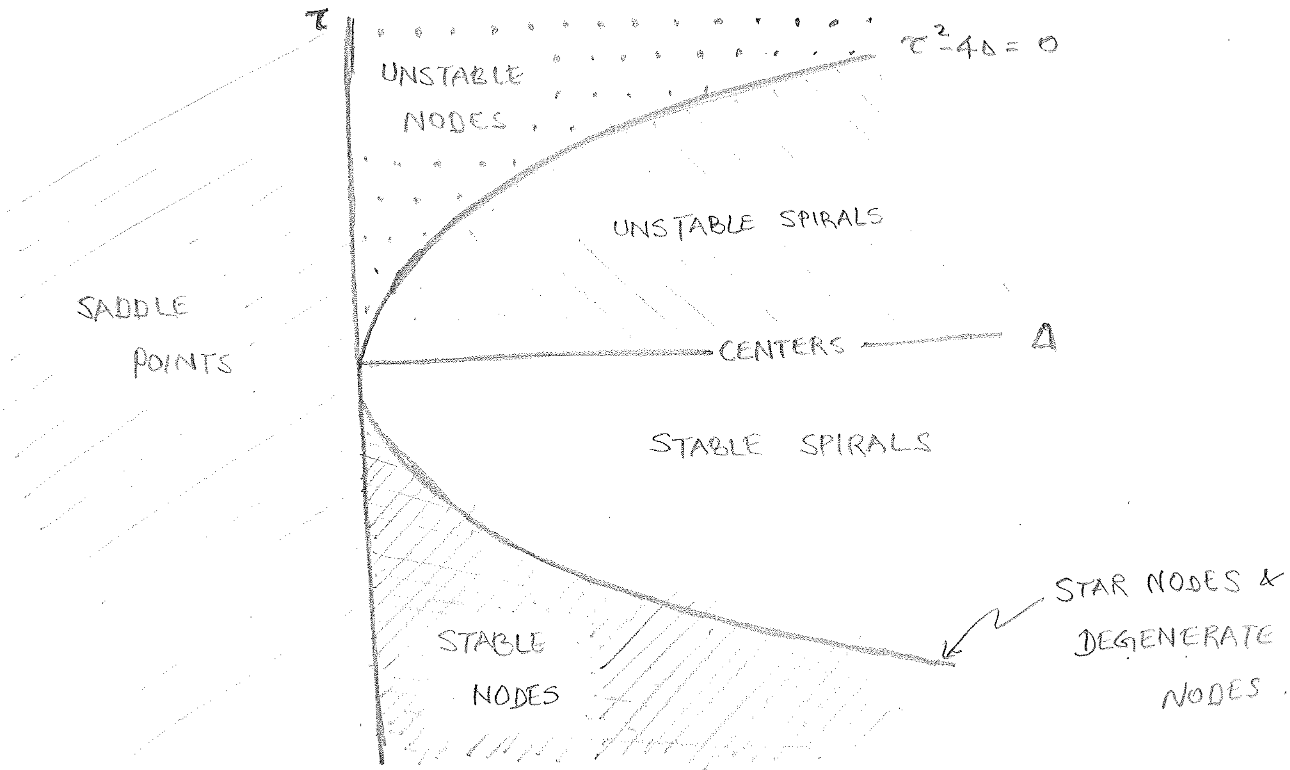


FP $\begin{bmatrix}0\\0\end{bmatrix}$ is called a <u>CENTER</u>



FP $\begin{bmatrix}0\\0\end{bmatrix}$ is called an <u>UNSTABLE FOCUS / SPIRAL</u>

<u>CLASSIFICATION OF FIXED POINTS (again) IN 2D</u>



DRAWING THE PHASE PORTRAIT FOR NONLINEAR SYSTEMS

Essentially the general way to draw the phase portrait for a nonlinear system $\dot{Y} = f(Y)$ is numerical – that is, solve the ODE for different initial conditions and plot $y_1(t)$ vs $y_2(t)$.

- But one can <u>sometimes</u> deduce the qualitative structure of the phase portrait by finding all the fixed points of the nonlinear system, determine the eigenvalues (& eigenvectors) corresponding to

Let us consider an example: <u>Simple pendulum with small c</u>.

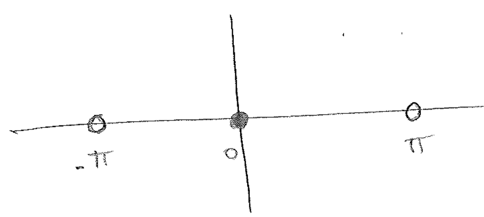
Equation: $\ddot{\theta} + \frac{g}{l} \sin\theta + c\dot{\theta} = 0$

$$y_1 = \theta \qquad \dot{y}_1 = y_2$$
$$y_2 = \dot{\theta} \qquad \dot{y}_2 = -k \sin y_1 - c y_2 \qquad \text{where } k = g/l$$

<u>FIXED POINTS</u>

$$y_2 = 0$$
$$\sin y_1 = 0 \Rightarrow y_1 = 0, \; y_1 = n\pi$$

Let us consider the 3 fixed points $y_1 = 0$, $y_1 = \pi$, and $y_1 = -\pi$.



$y_1 = 0$ is the <u>stable</u> equilibrium

$y_1 = \pm\pi$ are two different representation of the inverted <u>unstable</u> equilibrium.

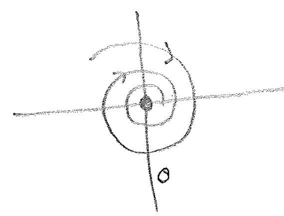
LINEARIZATION AND LINEAR STABILITY

We have looked at this before:

At $(0,0)$, the Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}$

We know that the eigenvalues are $\lambda_{1,2} = \dfrac{-c \pm \sqrt{c^2 - 4k}}{2}$

For sufficiently small $\underbrace{c}_{\text{(damping)}}$, these eigenvalues will be complex conjugates and the fixed point will be a STABLE SPIRAL.



At $(\pi, 0)$, the Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix}$

The eigenvalues are $\lambda_{1,2} = \dfrac{-c \pm \sqrt{c^2 + 4k}}{2}$.

$\underset{<0}{\tau = -c}$, $\underset{<0}{\Delta = -k}$, $\tau^2 - 4\Delta > 0 \Rightarrow$ SADDLE POINT (UNSTABLE).

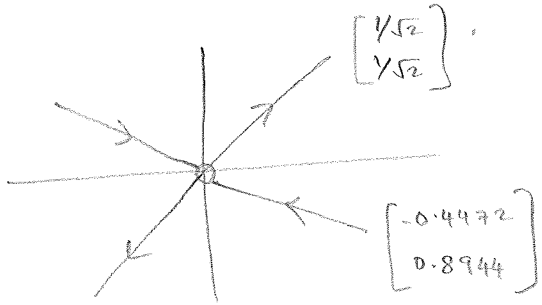
To calculate eigenvectors let us make the parameters precise. for simplicity

$c = 1$, $k = 2$, $\lambda_1 = \dfrac{-1 + \sqrt{9}}{2} = 1$, $\lambda_2 = \dfrac{-1 - \sqrt{9}}{2} = -2$.

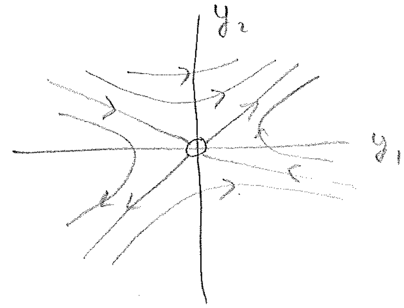
Eigenvectors are respectively (using MATLAB). $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix}$

UNSTABLE DIRECTION

STABLE DIRECTION

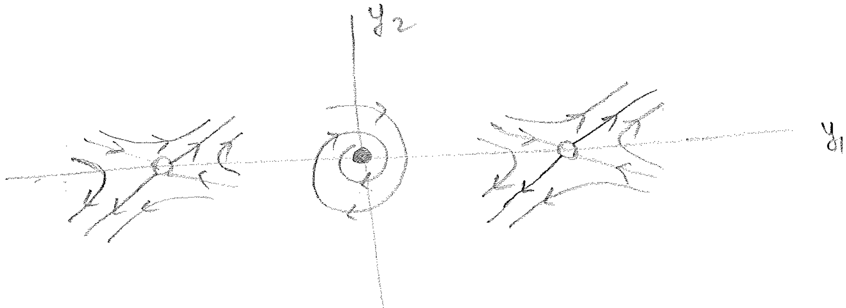


STABLE AND UNSTABLE MANIFOLD FOR SADDLE POINT.



PHASE PORTRAIT NEAR $(\pm\pi, 0)$

PUTTING THE PICTURES TOGETHER



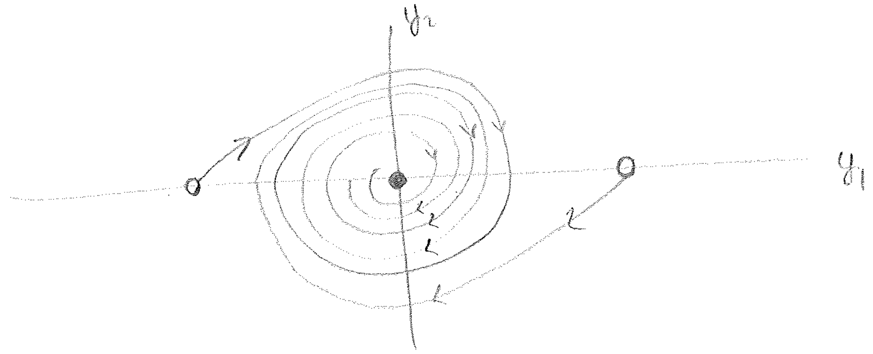
So far so good. We can get a picture like above by following a sequence of well-defined steps.

But to complete the picture, we have no general technique that always works, except for computing the phase portrait by solving the ODE (say numerically).

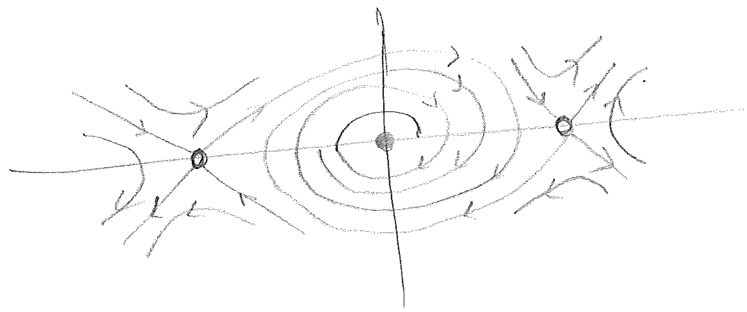
In this particular case, we can do a little better by appealing to physical intuition and knowledge of the nature of the solutions.

- If you start from the inverted position $(\pm\pi, 0)$ and give it an arbitrarily small perturbation, the pendulum oscillates and eventually comes to rest in the stable bottom position $(0,0)$.

In other words, the saddle points are connected to the stable spiral.



CONNECTING THE SADDLES TO THE SPIRAL (TWO DIFFERENT SPIRALS).



ADDING THE REST OF THE SADDLE.

Bifurcations (Intro / Rough outline)

Recall from earlier

Colloquially, a bifurcation is a qualitative change in the dynamics of a system when some system parameter is changed. More formally, a bifurcation is said to have occurred when the phase portrait's "topological structure" changes when a parameter is varied.

Example: change in the number and stability of fixed points.

The following are some of the most common bifurcations in 2D dynamics, very analogous to the corresponding 1D bifurcations.

In the following examples, you will notice that all the "bifurcation action" is happening along the x direction, whereas in the y direction, there is only simple stable dynamics.

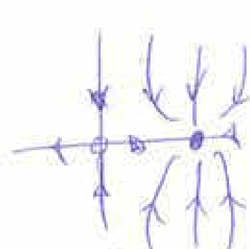
Thus, it may superficially seem that these are 'special examples.'

But they are not really that special. Rather, they are called 'normal forms' or 'canonical forms'. Any other dynamical system having a similar bifurcation can be transformed, at least locally, into these normal forms using a continuous (often differentiable) transformation of variables. That is, these simple examples (normal forms) have the same topological structure of every such bifurcation (locally).

Bifurcations in 2D

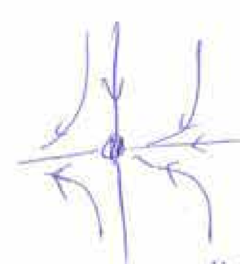
Saddle node:

$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$



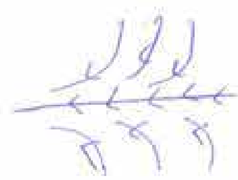
saddle & node

$\mu > 0$



$\mu = 0$

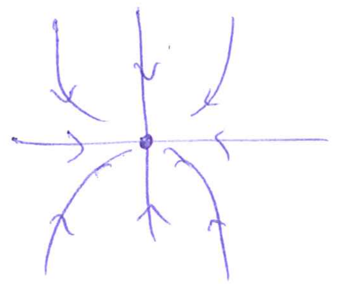
a saddle point and a node come together and vanish as μ is varied across $\mu = 0$.



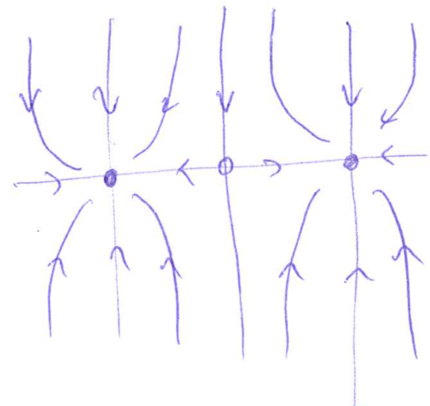
$\mu < 0$ (no FPs)

Supercritical pitchfork

$\dot{x} = \mu x - x^3 \quad \dot{y} = -y$



becomes



$\mu < 0$

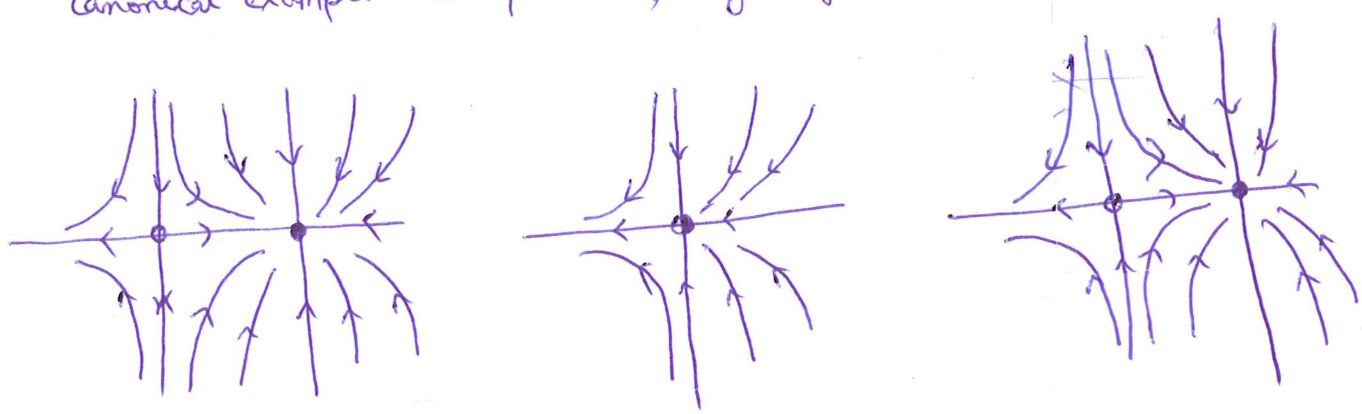
(one stable node).

$\mu > 0$. (2 stable nodes + 1 saddle point.)

As μ is change, one node becomes 3 FPs, two nodes + 1 saddle.

Transcritical bifurcation

Canonical example $\dot{x} = \mu x - x^2, \quad \dot{y} = -y$



$\mu < 0$

$\mu > 0$

two fixed pts

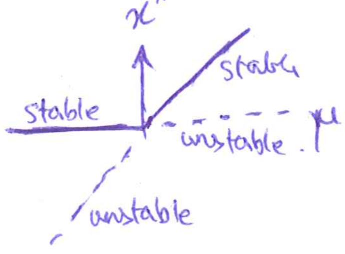
$x^* = 0$ – stable

$x^* = \mu < 0$ – unstable

two fixed points

$x^* = 0$ – unstable

$x^* = \mu > 0$ stable.



$x^* = 0$ goes from stable to unstable as μ is changed

and at $\mu = 0$ the phase portrait is topologically different from when $\mu > 0$ or $\mu < 0$.

Hence this is a bifurcation.

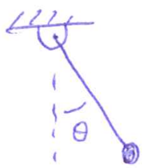
(even though the $\mu > 0$ and $\mu < 0$ portraits look similar)

Bifurcations in HW2 problem of stabilizing an inverted pendulum with a torsional spring

BIFURCATIONS EXAMPLE

HW2

Q3



damped simple pendulum

$$\ddot{\theta} + b\dot{\theta} + \frac{g}{L}\sin\theta = 0$$

Two fixed points (essentially) $\theta^* = 0$ and $\theta^* = \pi$

with rest position $\theta = \pi$ and stiffness k.

$$\ddot{\theta} + b\dot{\theta} + \frac{g}{L}\sin\theta + k(\theta - \pi) = 0$$

(a) At $\theta = \pi$, $k > g/L$ for stability. g/L is the local "negative stiffness" due to gravity - so k the positive stiffness due to the spring needs to be high enough that $k - g/L > 0$.

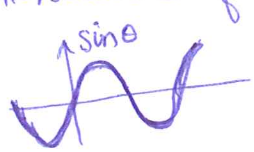
For $k > g/L$, $\theta = \pi$ is the only fixed point (as can be seen from the first picture on the next page).

(b) What happens to the (number of) fixed points as the stiffness k is reduced?

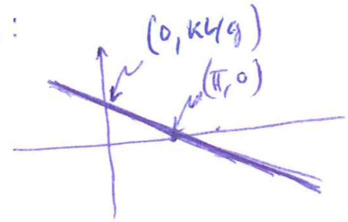
Fixed point when $g/L \sin\theta + k(\theta - \pi) = 0$

(or) $\frac{g}{L}\sin\theta = k(\pi - \theta)$. intersections of two curves:

$\sin\theta = \frac{kL}{g}(\pi - \theta)$



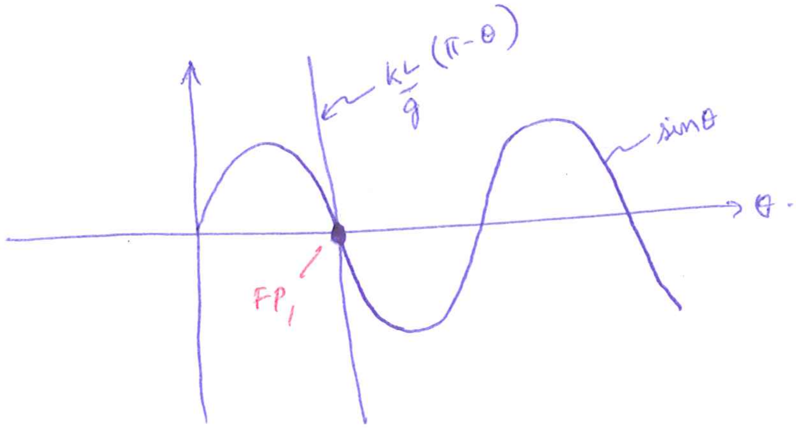
and



Fixed points & stability as k is decreased.

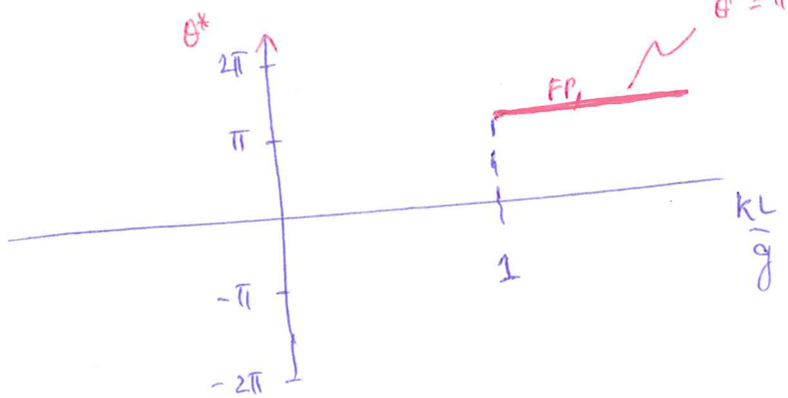
sufficiently stiff limit $k > g/L$. Only 1 FP. at $\theta = \pi$.

$\frac{kL}{g} > 1$



when $kL/g > 1$, the slope of $\frac{kL}{g}(\pi-\theta)$ is greater than the slope of $\sin\theta$.

Bifurcation Diagram so far.

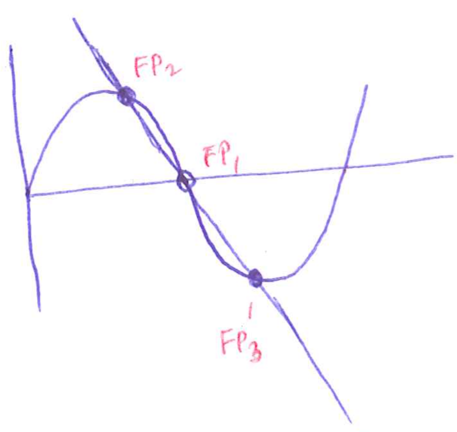


$\theta^* = \pi$ is the only stable FP for $kL/g > 1$

k just less than g/L

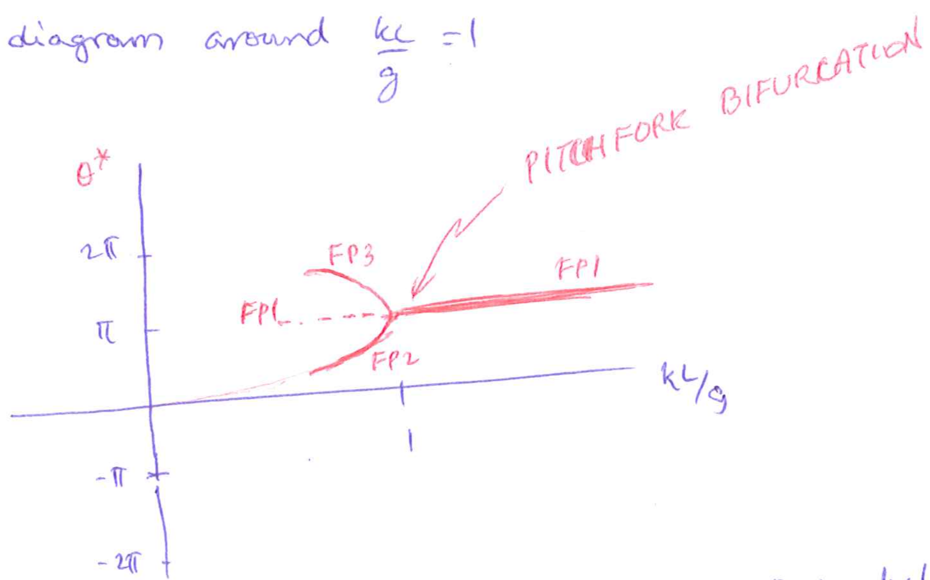
(or $\frac{kL}{g}$ just less than 1)

- There are 3 intersections $\Rightarrow$ 3 FPs.
- one unstable and 2 stable.



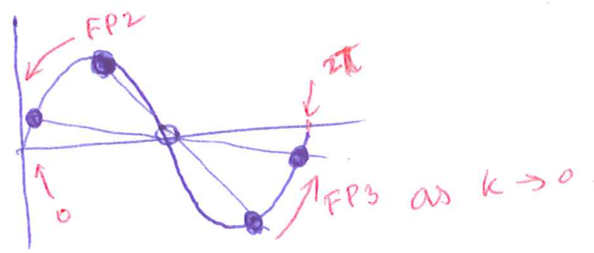
(How to know that FP1 is unstable and FP2 and FP3 are stable? See end of this discussion on page 6.)

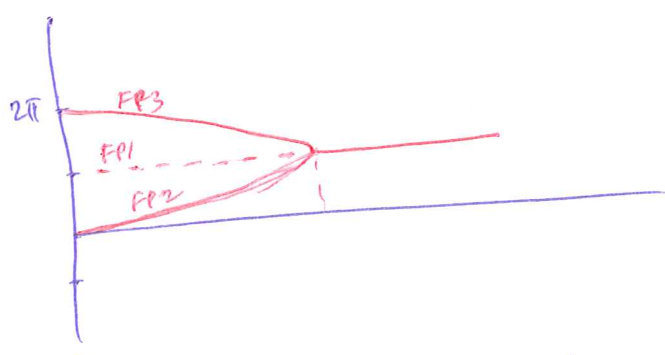
Bifurcation diagram around $\frac{kL}{g} = 1$



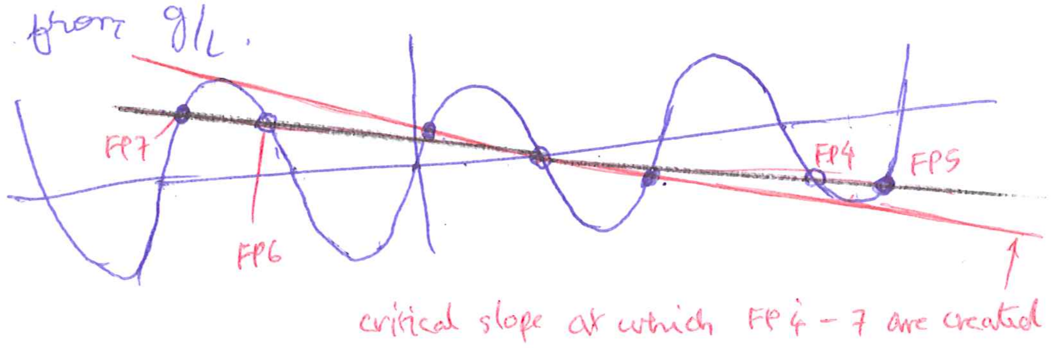
As k is reduced other fixed points arise. But before that let's see what FP2 and FP3 do as $k \to 0$.

We see that FP2 → 0 and FP3 → 2π.





Let us now consider how other fixed points arise as k is reduced from g/L.



The red line is when there are exactly 5 FPs.

for K < redline slope (blackline) ⇒ 7 FPs.

for K > redline slope (⇒ 3 FPs

So just at the redline slope (& below it), 4 new FPs are created · in 2 pairs.

- call them FP4 – FP7 as in previous page.
- FP4 & FP6 are <u>roughly</u> located at $+3.5\pi$ when they are created.
- FP6 & FP7 are <u>roughly</u> located at -1.5π when created.

critical red slope is <u>roughly</u> $\sim \frac{1}{2.5\pi} = \frac{kL}{g}$.

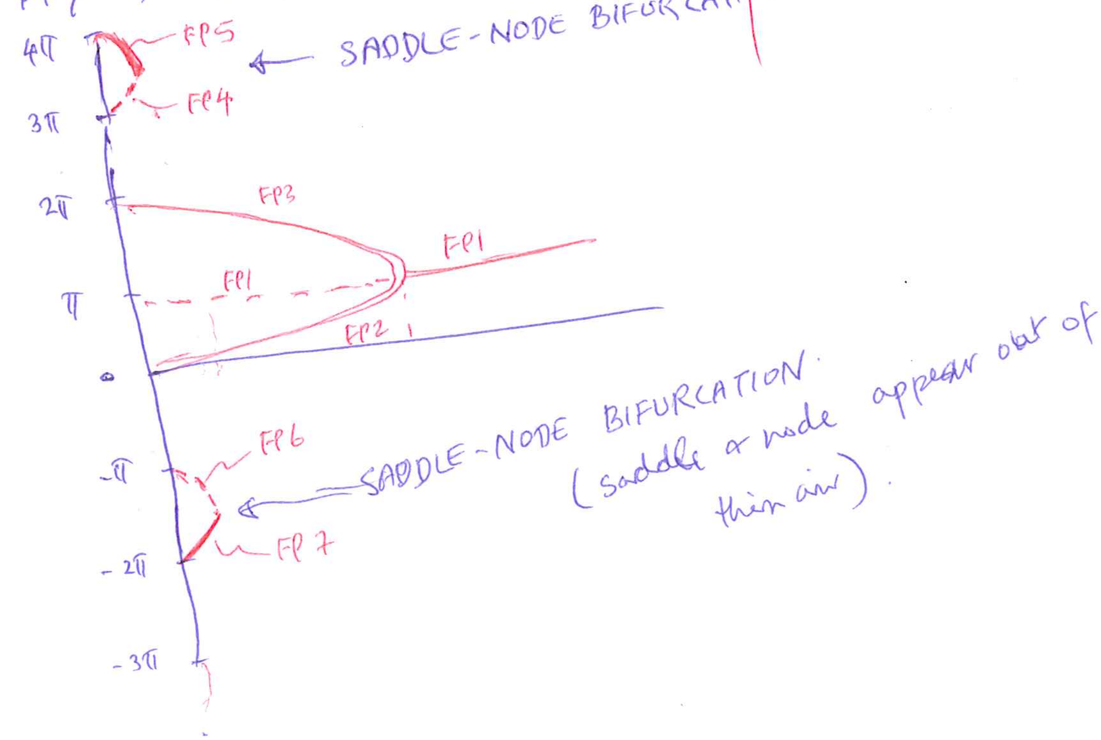
- when $K \to 0$

$FP4 \to 3\pi$

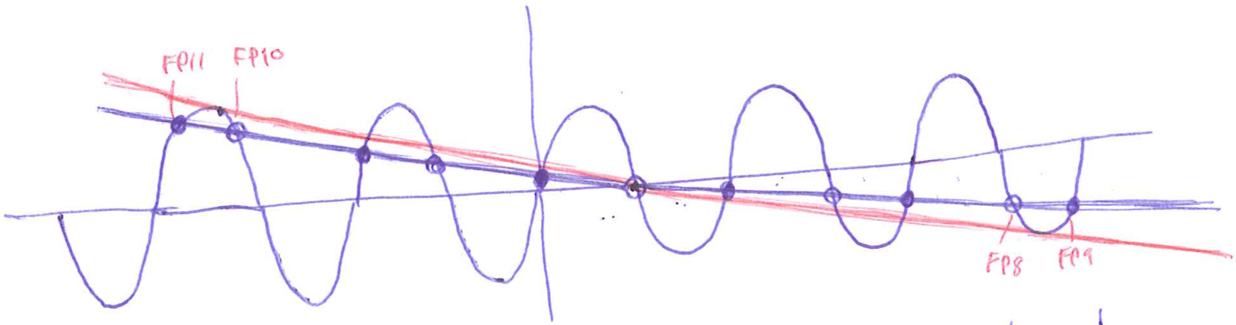
$FP5 \to 4\pi$

$FP6 \to -\pi$

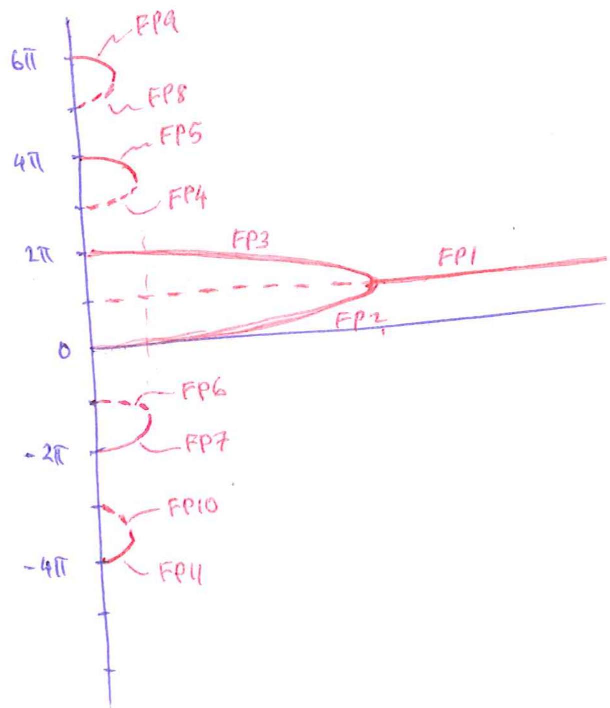
$FP7 \to -2\pi$,



As k is reduced even further just less than another critical slope (shown in red below).
there arise 4 more fixed points, again in 2 pairs
(one pair with $\theta > 0$ and another pair with $\theta < 0$).



FP8 and FP9 appear in a saddle-node bifurcation
So do FP10 and FP11.
So extending the Bifurcation diagram a bit:



Note that as $k \to 0$
FP8 $\to 5\pi$
FP9 $\to 6\pi$
FP10 $\to -3\pi$
FP11 $\to -4\pi$

As k is decreased further, we get more and more fixed points, all arising in pairs of saddle-node bifurcations.

Ok, how did we know what the stability of the fixed points were?

<u>FACT</u> (you should be able to show this) (Exercise). : The fixed points of

$$\ddot{x} + b\dot{x} + f(x) = 0$$

and $b\dot{x} + f(x) = 0$

are the <u>same</u> and <u>HAVE THE SAME STABILITY</u>.

Therefore, the fixed points and stability of $\ddot{\theta} + b\dot{\theta} + \frac{g}{L}\sin\theta + k(\theta - \pi) = 0$

are the same as the fixed points and stability of

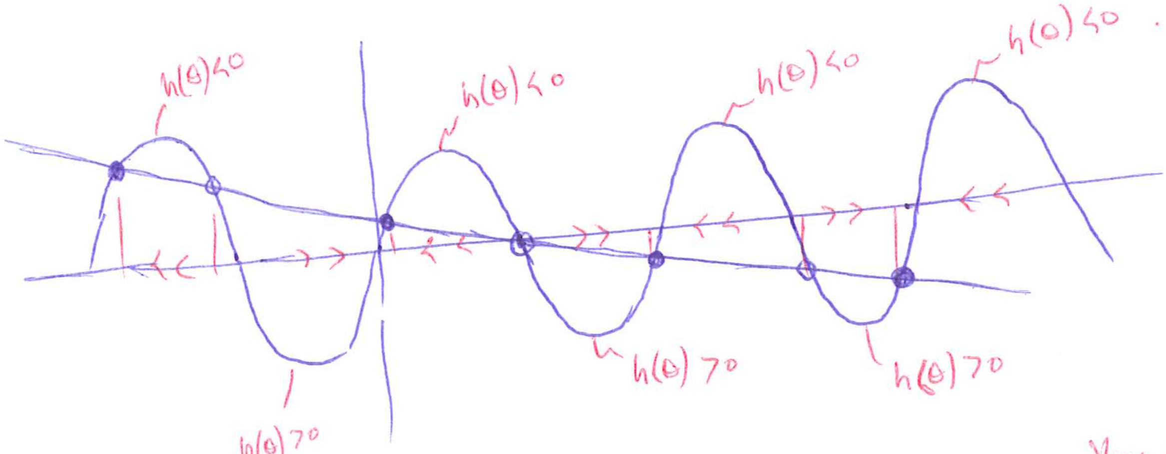
$$b\dot{\theta} + \frac{g}{L}\sin\theta + k(\theta - \pi) = 0$$

So we can analyze the stability of the 1D equation:

$$b\dot{\theta} = k(\pi - \theta) - \frac{g}{L}\sin\theta = h(\theta) \text{ say.}$$

$h(\theta) < 0$ if $\frac{g}{L}\sin\theta > k(\pi - \theta)$

$h(\theta) > 0$ if $\frac{g}{L}\sin\theta < k(\pi - \theta)$.



← plot determining stability.

You can do a full 2D analysis to show that the fixed points are saddles, etc.