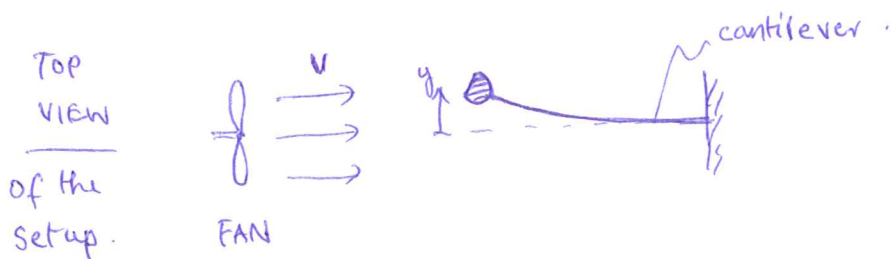
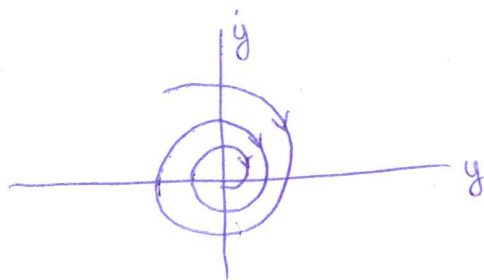


LIMIT CYCLES      INTRODUCTION

We start with a demonstration of an "aeroelastic" instability and oscillation.



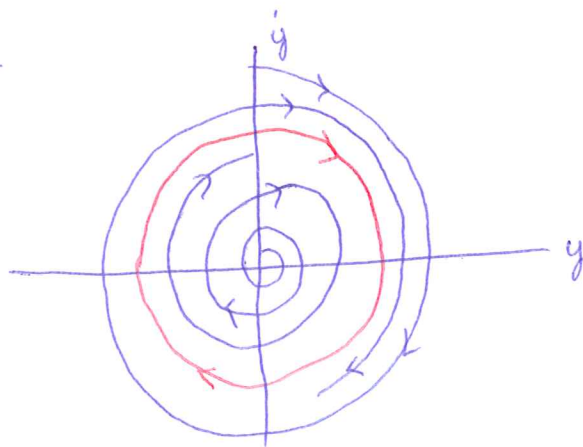
With the fan switched off ( $v=0$ ), the cantilever beam is essentially a damped oscillator. So the phase portrait looks like



$(0,0)$  = Fixed point  
STABLE SPIRAL.  
(underdamped oscillations)

When the fan is turned on ( $v$  large), the cantilever beam starting from  $(0,0)$  reaches a steady (apparently periodic) oscillation. That is,  $(0,0)$  has become an unstable spiral, and there exists a periodic motion with finite amplitude. (shown in red below).

The phase portrait now looks like:



Large initial conditions decay.  
Small initial amplitudes grow.  
With an asymptotically stable periodic orbit in between.

The periodic orbit here is called a limit cycle.

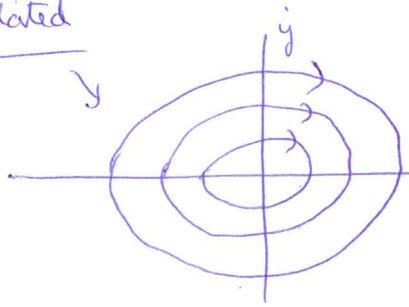
Formally, a limit cycle is defined as an "isolated" periodic orbit. Limit cycles can be stable or unstable just like FPs.

What does "isolated" mean?

"Isolated" means that arbitrarily close to our limit cycle there exists no other periodic orbit.

For instance, the periodic orbits of a simple harmonic oscillator are

NOT isolated



$$m\ddot{y} + ky = 0$$

$(0,0)$  is a CENTER. - there is a continuum of periodic orbits.

That is, arbitrarily close to any given periodic orbit, there is another periodic orbit. (There is a periodic motion at every amplitude).

Linear systems cannot have limit cycles. Only nonlinear systems can.

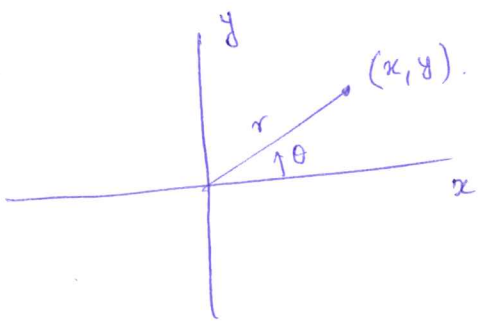
Toy example for limit cycles

(Strogatz).

Dynamics in polar coordinates

$$\dot{r} = r(1-r^2)$$

$$\dot{\theta} = 1$$



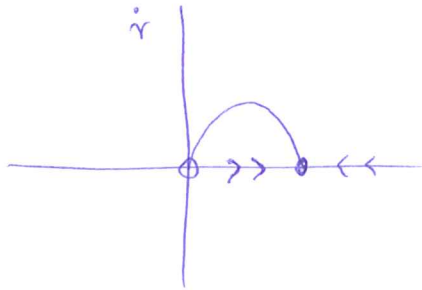
$$x = r \cos \theta$$

$$y = r \sin \theta$$

$r$  and  $\theta$  are decoupled, so can be studied independently.

3

FPs for  $r$ -dynamics.  $r^* = 0$   
and  $r^* = 1$ .



$r^* = 1$  is a stable fixed point.

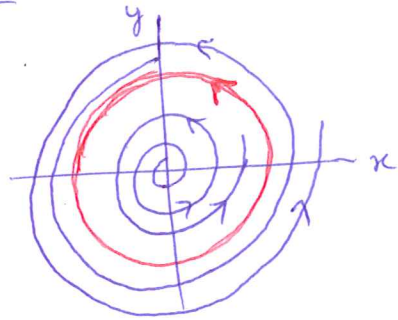
So if  $r(0) < 1$ , then  $r$  increases  $\rightarrow 1$ .

if  $r(0) > 1$ , then  $r$  decreases to 1.

if  $r(0) = 1$ ,  $r(t) \equiv 1$  for all  $t$ .

$\theta$  dynamics  $\dot{\theta} = 1 \Rightarrow \theta(t) = t$ .  $\theta(t)$  increases monotonically with  $t$ .

phase portrait - (useful to think about  $r < 1$ ,  $r = 1$ ,  $r > 1$ ).



Limit cycle at  $r \equiv 1$  and  $\dot{\theta} = 1$ .  
circle of radius 1. in the  
 $x$ - $y$  plane.

$(0,0)$  is an unstable spiral.  
because if  $r < 1$ ,  $r$  increases  
to 1.

Why are the above trajectories anti-clockwise?

because  $\dot{\theta} = 1$  as given and  $\theta$  is measured anti-clockwise  
by convention in polar coordinates.

## Van der Pol equation

This is a famous example for limit cycles, due to Dutch electrical engineer Balthasar van der Pol (1930s) initially as a model for some electrical circuits involving vacuum tubes.

Since then the equation has been used in a variety of settings notably in neuroscience.

Equation:  $\ddot{x} + \underbrace{\mu(x^2-1)}_{\text{nonlinear damping term}} \dot{x} + x = 0$   $\mu > 0$

$\mu(x^2-1)$  is like a (nonlinear) damping coefficient.

When  $|x| < 1$ ,  $\mu(x^2-1)$  - negative damping coefficient  
So we might expect instability and growth for  $|x| < 1$ , i.e. small amplitudes.

When  $|x| > 1$ ,  $\mu(x^2-1)$  - positive damping coefficient  
So we might expect large amplitudes + velocities to decay.

These observations suggest (but do not prove) that there might be a stable limit cycle at an intermediate amplitude.

And indeed there is.

Note: For  $\mu = 0$ , the equation becomes  $\ddot{x} + x = 0$  undamped mass-spring system which has a continuum of periodic orbits.

For  $\mu > 0$ , but arbitrarily close to zero, there is a stable limit cycle!  
This is also an example of a singular perturbation (in which  $\mu = 0$  is qualitatively different from  $\mu \rightarrow 0$ ,  $\mu \neq 0$ ).

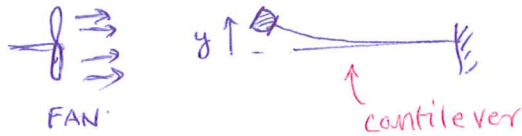
# LIMIT CYCLES - AEROELASTIC OSCILLATION EXAMPLE

## References

①

+ HOPF BIFURCATIONS, etc.

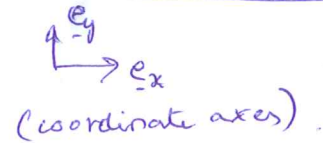
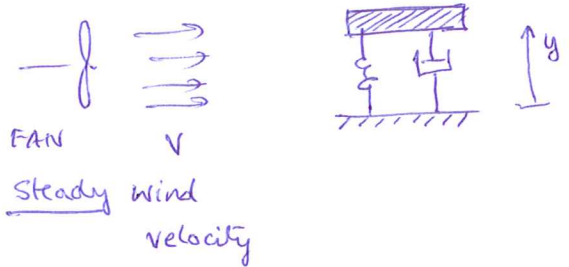
Live Demo setup (top view)



Article: The square prism as an aeroelastic nonlinear oscillator. G.V. Parkinson & J.D. Smith. *Quant. Jour. Mech & Applied Mech.*, Vol XVII, Pt 2, 1964, p 225-239.

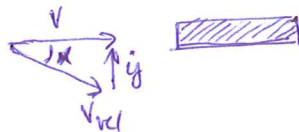
Book: *Nonlinear Dynamics and Chaos*. J.M.T. Thompson and H.B. Stewart. pages 58-61.

→ Simplified version



Wind velocity relative to the body

$$\underline{v}_{rel} = v \underline{e}_x - \dot{y} \underline{e}_y$$



$\alpha$ . angle of attack.  $\sim \dot{y}/v$

Let  $F_y$  be the force of the wind on the body along  $-\underline{e}_y$ .  $F_y$  depends not only on the wind speed  $v$  but also the body velocity  $\dot{y}$ . In particular, the force  $F_y$  is given by

the formula:

$$F_y = \frac{1}{2} C_{Fy} \rho v^2 a \quad (2)$$

where  $\rho$  is the density of air.

$v$  is the wind speed.

$a$  is cross-sectional area  $\perp$  to wind direction.

$C_{Fy}$  "drag coefficient" for  $F_y$ . (a mixture of the normal drag & lift coefficients)

which is a function of  $\alpha$ , or equivalently,  $(\dot{y}/v)$ .

$$m\ddot{y} + b\dot{y} + ky = F_y. \quad (1)$$

By symmetry, we must have

$$C_{Fy} \left( \frac{\dot{y}}{v} \right) = - C_{Fy} \left( \frac{\dot{y}}{v} \right) \quad - (3)$$

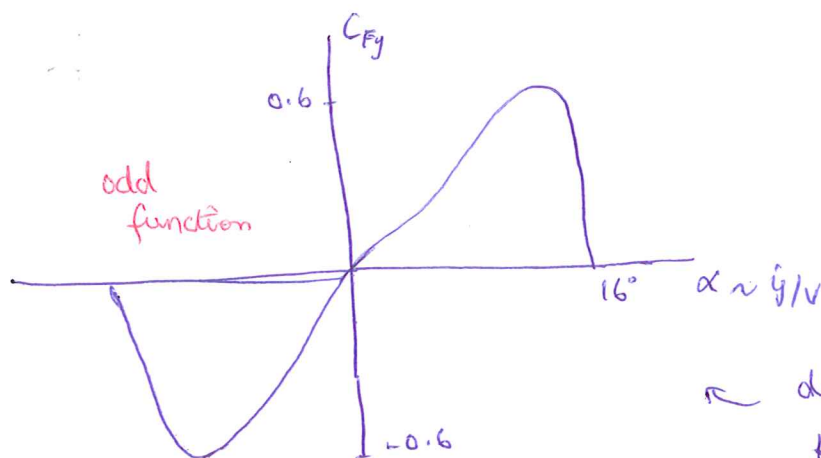
or equivalently  $F_y \left( -\frac{\dot{y}}{v} \right) = - F_y \left( \frac{\dot{y}}{v} \right)$

(2) If force is along  $\underline{e}_y$  for positive  $y$ , the force is equal and opposite for negative  $y$ .

That is,  $C_{Fy}$  is an odd function of  $\frac{\dot{y}}{v}$ .

So we may write

$$C_{Fy} = A_1 \left( \frac{\dot{y}}{v} \right) - A_2 \left( \frac{\dot{y}}{v} \right)^3 + A_3 \left( \frac{\dot{y}}{v} \right)^5 - A_4 \left( \frac{\dot{y}}{v} \right)^7 \quad - (4)$$



$$\begin{aligned} A_1 &= 2.69 \\ A_2 &= 168 \\ A_3 &= 6270 \\ A_4 &= 59900 \end{aligned}$$

↑ Parkinson & Smith (1964)

↖ determined empirically from wind tunnel experiments with steady wind. So Equ (4) + (2) is a "quasisteady" approximation for  $F_y$ .

Writing the equation for the spring-mass-damper system with the wind force, we have

$$m\ddot{y} + b\dot{y} + ky = \frac{1}{2} \rho v^2 a_1 \left[ A_1 \left( \frac{\dot{y}}{v} \right) - A_2 \left( \frac{\dot{y}}{v} \right)^3 + A_3 \left( \frac{\dot{y}}{v} \right)^5 - A_4 \left( \frac{\dot{y}}{v} \right)^7 \right]$$

Rearranging,

$$m\ddot{y} + ky + \dot{y} \left[ b - \frac{1}{2} \rho a V A_1 \right] - \frac{1}{2} \rho V^2 a \left[ -A_2 \left( \frac{\dot{y}}{V} \right)^3 + A_3 \left( \frac{\dot{y}}{V} \right)^5 - A_4 \left( \frac{\dot{y}}{V} \right)^7 \right] = 0$$

linear damping term

wind force with nonlinear dependence on  $\dot{y}$ . (no linear term).

Let us think about the dynamics in the  $y-\dot{y}$  phase plane.

Clearly  $(0,0)$  is a fixed point.

It can be shown that  $(0,0)$  is a stable fixed point

if  $b - \frac{1}{2} \rho a V A_1 > 0$  (positive linear damping)

and unstable fixed point if

$b - \frac{1}{2} \rho a V A_1 < 0$  (negative linear damping coefficient)

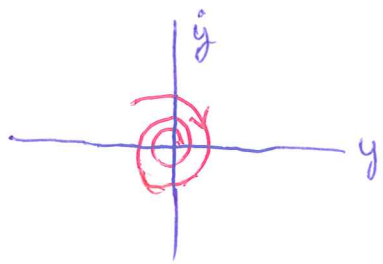
you can use Jacobians to show this if you want

In particular, as  $b - \frac{1}{2} \rho a V A_1$  goes from  $> 0$  to  $< 0$ , the fixed point  $(0,0)$  goes from being a stable SPIRAL to an unstable SPIRAL.

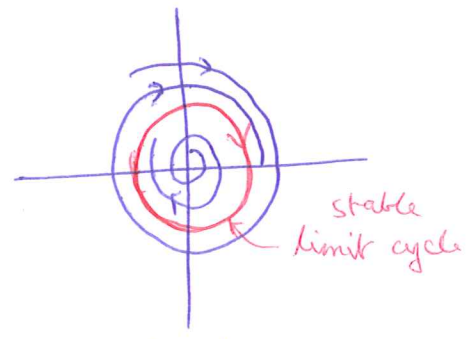
[It is a SPIRAL as opposed to a node because the effective damping is  $b - \frac{1}{2} \rho a V A_1$  and we are considering a situation when this effective damping is close to zero]

It turns out that in addition to  $(0,0)$  becoming unstable, a small limit cycle is born around  $(0,0)$  when

$$b - \frac{1}{2} \rho a v A_1 \text{ becomes } < 0.$$



$$b - \frac{1}{2} \rho a v A_1 > 0$$



$$b - \frac{1}{2} \rho a v A_1 < 0$$

$$\Rightarrow v < \frac{2b}{\rho a A_1}$$

$$v > \frac{2b}{\rho a A_1}$$

Critical wind speed  $v_{co} = \frac{2b}{\rho a A_1}$

Below this critical wind speed ( $v < v_{co}$ ),  $(0,0)$  is stable - no oscillations.

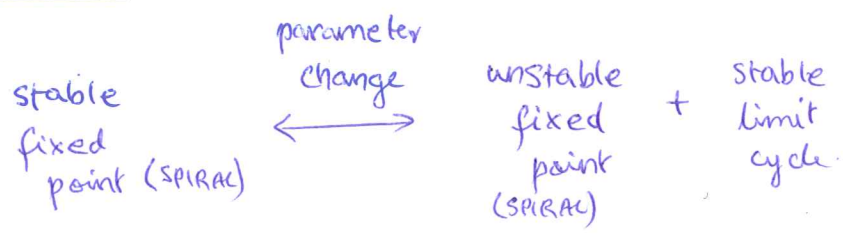
Above this critical wind speed ( $v > v_{co}$ ),  $(0,0)$  is unstable and a limit cycle oscillation is stable.

This is a "Hopf Bifurcation". In particular, it is called a supercritical Hopf bifurcation (in analogy with the supercritical pitchfork bifurcation) as the amplitude of the limit cycle grows continuously from zero. \* [when created, the limit cycle has zero radius and grows as  $v$  is increased above  $v_{co}$ ]

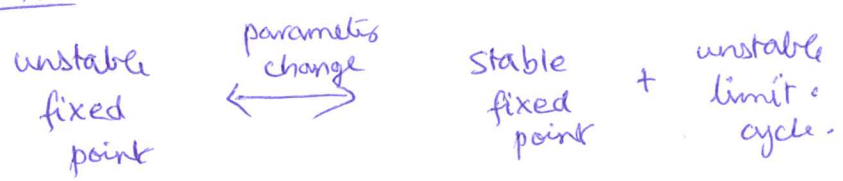
Hopf Bifurcation (Definition): The appearance or disappearance of a limit cycle by interaction with a fixed point that changes stability.



### Supercritical Hopf



### Subcritical Hopf

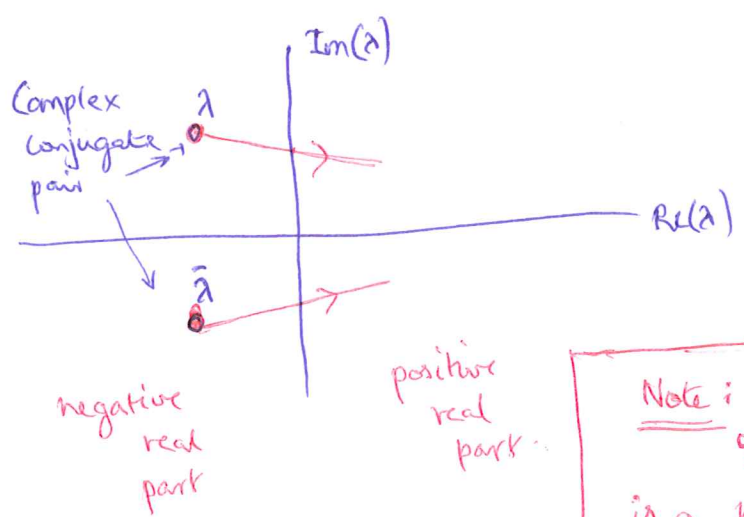


### Recognizing a Hopf Bifurcation

We can recognize a Hopf bifurcation by seeing what happens to the eigenvalues of the fixed point in question. The fixed point generally goes from being an unstable spiral to a stable spiral or vice versa.

So, the eigenvalues go from being complex conjugates with positive real parts to complex conjugates with negative real parts.

Motion of eigenvalues in the complex plane as some system parameter is varied.



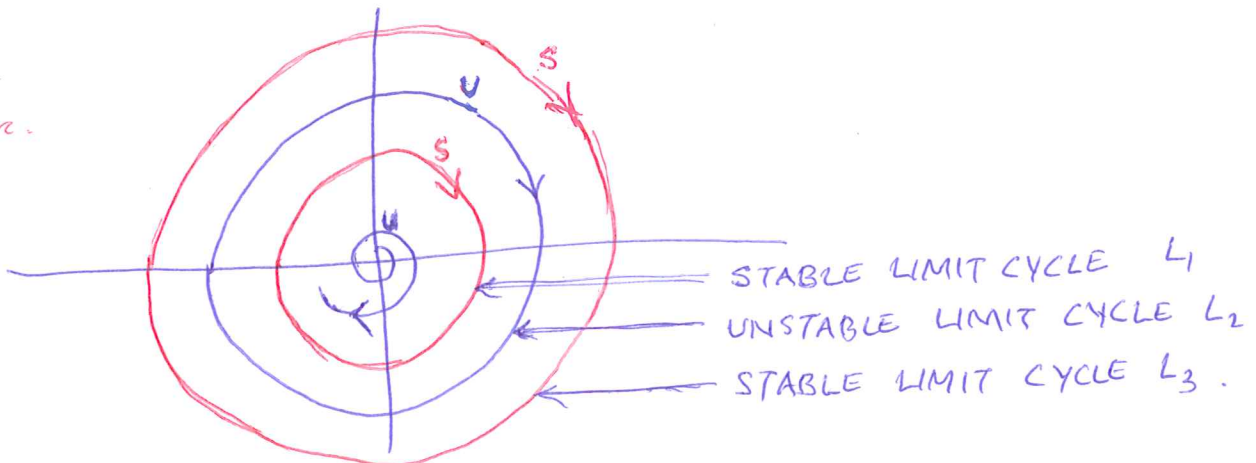
↑  
This is true in any dimension, not just 2. In N-dimensions, one complex conjugate pair of eigenvalues changes the sign of its real parts

Note: the changing of the sign of the eigenvalue real parts is a necessary condition for a Hopf bifurcation, BUT is not sufficient to predict limit cycles.

Coming back to the aeroelastic example, we can increase the velocity  $V$  higher and see what happens to the phase portrait.

As the wind speed  $V$  is increased, the system goes from having 1 limit cycle to 3 limit cycles!

(Note: This is just a qualitative picture. the shapes of the limit cycles are far from circular)



→ If you start from anywhere within  $L_1$ , you converge asymptotically to  $L_1$ .

→ If you start from anywhere between  $L_1$  and  $L_2$ , you converge asymptotically to  $L_1$ .

→ If you start from anywhere between  $L_2$  and  $L_3$ , you converge asymptotically to  $L_3$ .

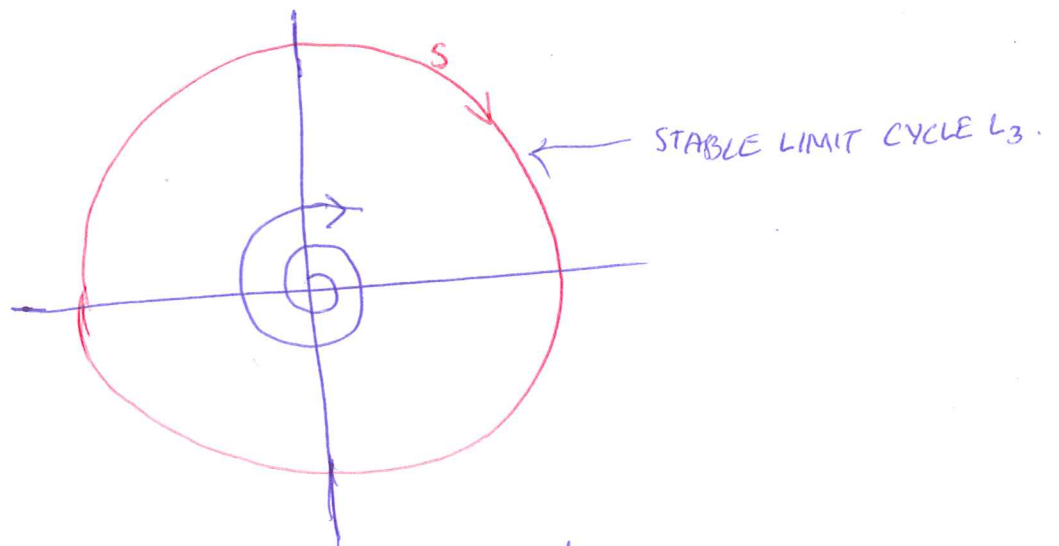
→ If you start from anywhere outside  $L_3$ , you converge asymptotically to  $L_3$ .

→ Start outside  $L_3$  and run time backward, you go to infinity

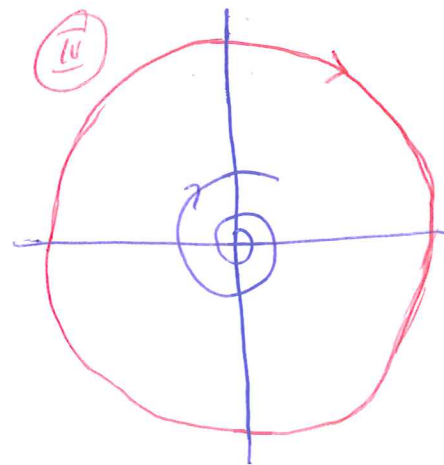
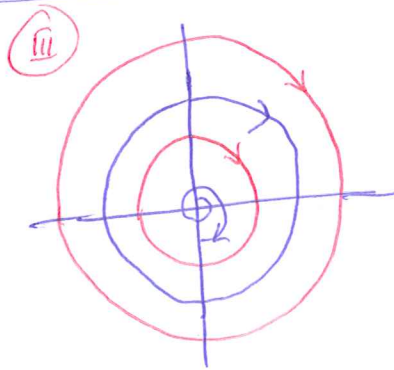
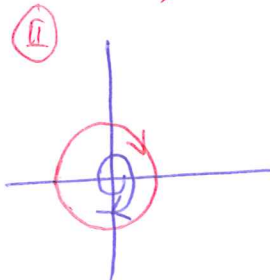
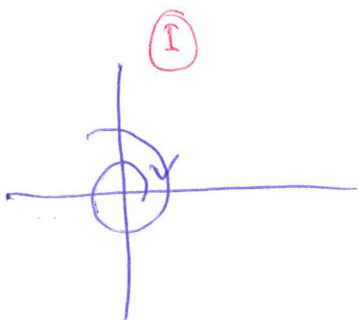
→ Start between  $L_1$  and  $L_3$ , and run time backward, you asymptotically approach  $L_2$ . (In 2D systems, unstable limit cycles become stable limit cycles when you run time backward. But this is NOT the case in general for N-Dim systems. Exercise: why?)

→ Start inside  $L_1$  and run time backward, you asymptotically approach the unstable fixed point  $(0,0)$ .

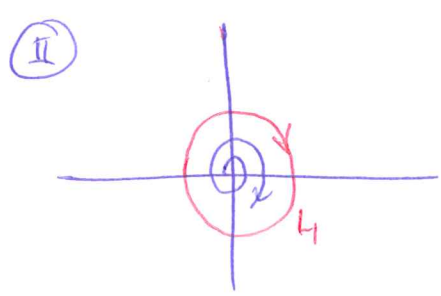
As the wind speed is increased even further, the limit cycles  $L_1$  and  $L_2$  come together and vanish, leaving only  $L_3$ .



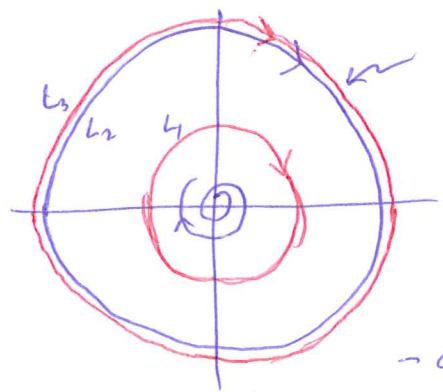
Sequence of phase portraits as  $v$  is increased.  
(qualitatively)



II  $\rightarrow$  III happens in the following manner (say at critical speed  $V_{c1}$ )



$V < V_{c1}$   
(just below)

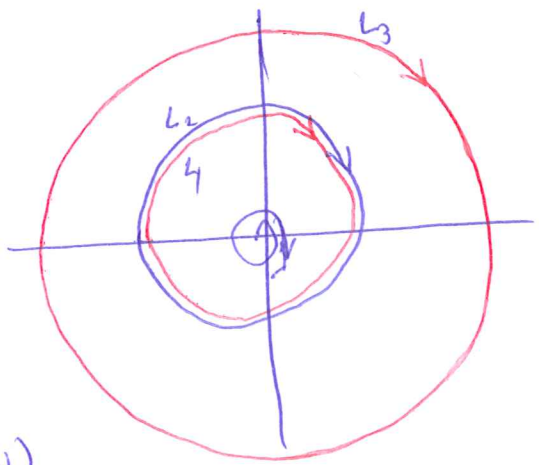


$V > V_{c1}$   
(just above)

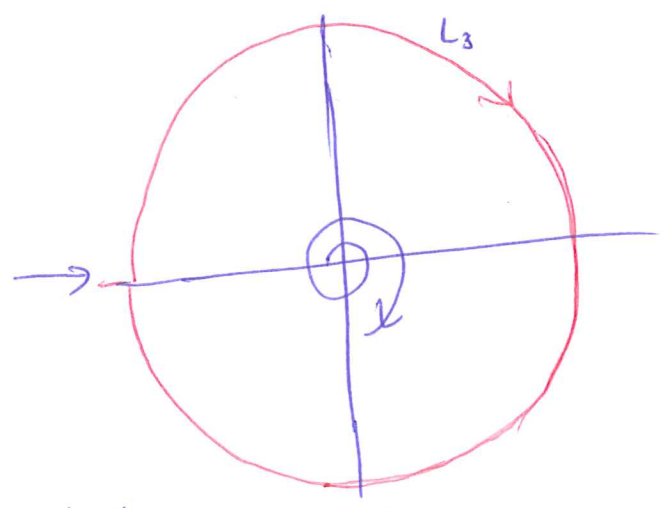
The stable and unstable limit cycles appear simultaneously - and they are arbitrarily close to each other (they become coincident at the critical velocity  $V_{c1}$ )

This bifurcation is an analog of the saddle-node bifurcation for fixed points - it is a saddle-node bifurcation of limit cycles - in which a stable and unstable limit cycle appear almost on top of each other.

III  $\rightarrow$  IV is also a saddle-node bifurcation of limit cycles.  $L_1$  and  $L_2$  come together and vanish, say at some critical velocity  $V_{c2}$ .

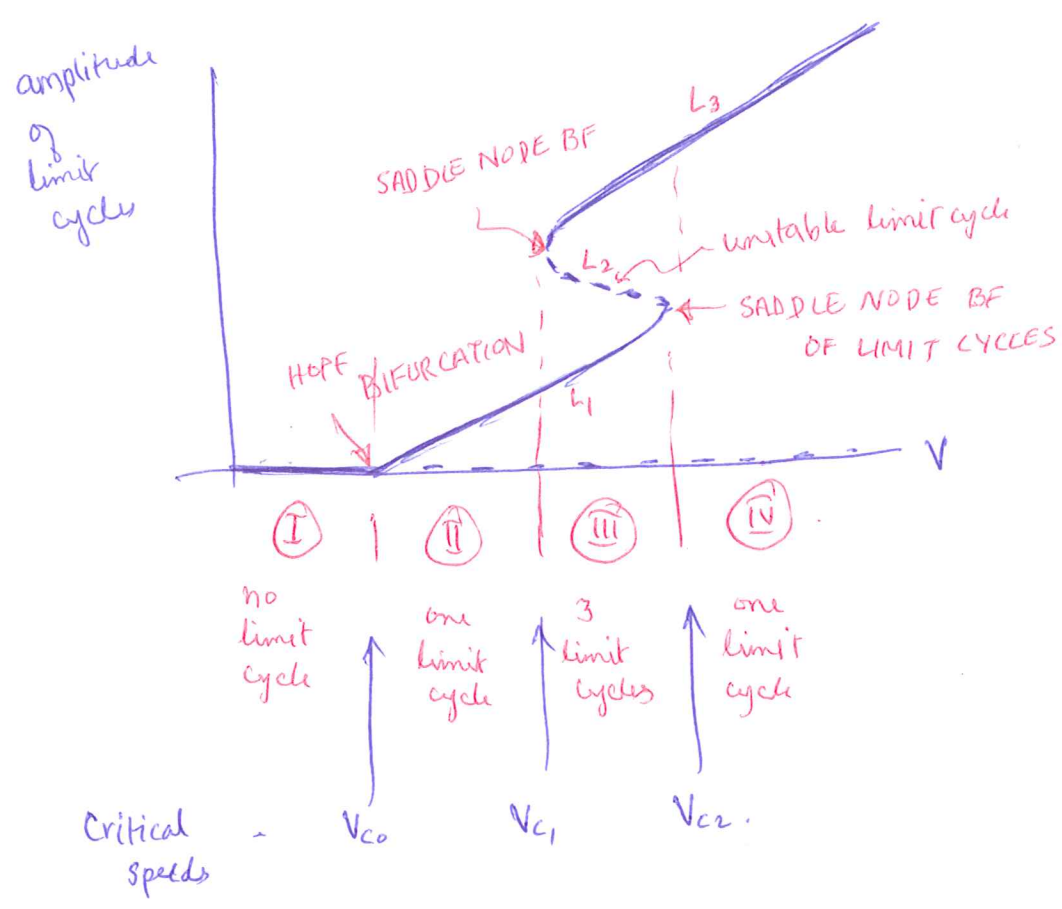


$V < V_{c2}$   
(just below)



$V < V_{c2}$ , (just above)

We can represent the sequence of qualitative changes that happen to the phase portrait by plotting the "amplitudes" of the limit cycle oscillations



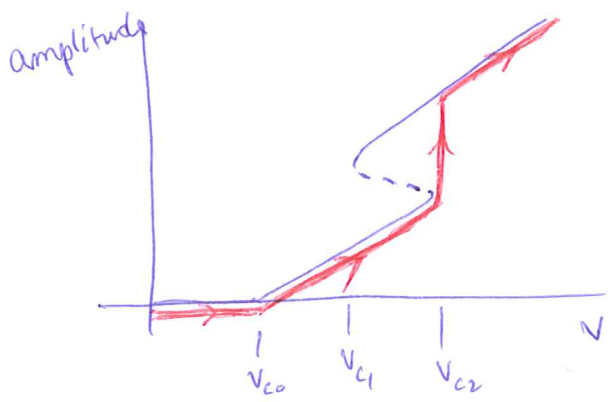
(the limit cycles are funny shaped so here by amplitude we mean "some" reasonable measure of size say max|x|)

The above plot captures aspects of the phase portrait for various fixed values of  $V$ .

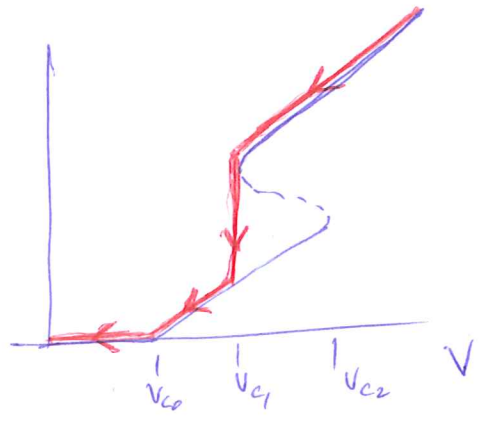
We can ask what the system behavior is when  $V$  is slowly increased as a function of time

(Note "slowly" is a key word. If  $V$  is increases very fast, there might be all kinds of transient effects associated with  $v(t)$ )

Increasing  $V(t)$



Decreasing  $V(t)$  from high  $V$



The system is started at rest  $(0,0)$   
As  $V$  is increased

until  $V_{c0}$ , the  $(0,0)$  is stable.  
After  $V_{c0}$ ,  $(0,0)$  is unstable, so the system starts oscillating - this is the stable limit cycle.  
As  $V$  is increased, the amplitude of the limit cycle increases.

After  $V_{c1}$ , two limit cycles are stable  $L_1$  and  $L_2$ . But the system is currently in  $L_1$  and there is no reason to go to  $L_2$  (unless there happens to be a huge perturbation).  
So the system remains in  $L_1$ .

After  $V_{c2}$ ,  $L_1$  vanishes, and the system goes to the ONLY stable structure available which is the large limit cycle.  
manifested as a jump in oscillation amplitude!

We start at high  $V$  and reduce  $V$ .  
The system starts at  $L_2$  the large limit cycle.  
As  $V$  is reduced less than  $V_{c2}$ , the system has 2 stable limit cycles  $L_1$  and  $L_2$ , but the system already in  $L_2$  has no reason to move to  $L_1$  - it continues to be in  $L_2$  as  $V$  is decreased further.

When  $V$  is reduced to less than  $V_{c1}$ , the limit cycle  $L_2$  suddenly vanishes and the system gets attracted to the only stable structure available, which is the limit cycle  $L_1$ . This is seen as a jump in oscillation amplitude downward.

The jump up in amplitude as  $V$  increases happens at  $V_{c2}$ .

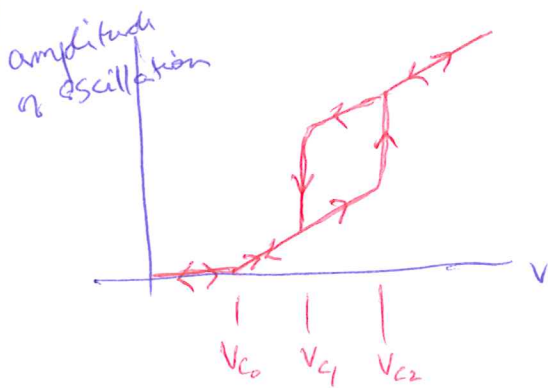
The jump down in amplitude as  $V$  decreases happens at  $V_{c1}$ .

The difference in oscillation behavior when  $V$  is increased versus when  $V$  is decreased is called "hysteresis" in oscillation amplitude.

(The word "hysteresis" is used when an experiment exhibits different behavior as a parameter is increased versus when the parameter is decreased slowly.)

(Note: The word "hysteresis" is also used in the context of inelastic (dissipative) material behaviors. But again, the word is used there because these materials exhibit different behavior (say  $\sigma$ - $\epsilon$  curves) when some parameter (say  $\epsilon$ ) is increased vs decreased).

Summary



The sudden jump in amplitude at  $V_{c2}$  can, in real systems, be unexpected and therefore sometimes catastrophic!

Suggestion: Use pplane to draw phase portraits at different velocity regimes. See what happens when the system is started at different initial conditions.

Exercise: Slowly vary  $V$  from zero in a simulation, increase + decrease and see what the oscillation amplitude is. Initial condition  $(0,0)$  + small perturbation.