DISCRETE DYNAMICAL SYSTEMS

By "discrete" here, we mean "discrete time".

(Discrete dynamical systems are interesting to study in their own right. But we introduce them here because they provide a formalism for understanding / analysing limit cycle stability, namely Poincaré maps)

Discrete Dynamical Systems are also called "maps".

Let us start with a simple example.

Eg. Take a piece of paper,

Say its thickness is x_0.

Fold it once. It's thickness doubles.

New thickness $x_1 = 2x_0$.

Thickness after n folds $= x_n$ say.

Then, thickness after $n+1$ folds $= x_{n+1} = 2x_n$.

$\boxed{x_{n+1} = 2x_n}$ — 1 $\quad x_n = 2^n x_0$

as $n \to \infty$, $x_n \to \infty$ (if you could keep folding the piece of paper)

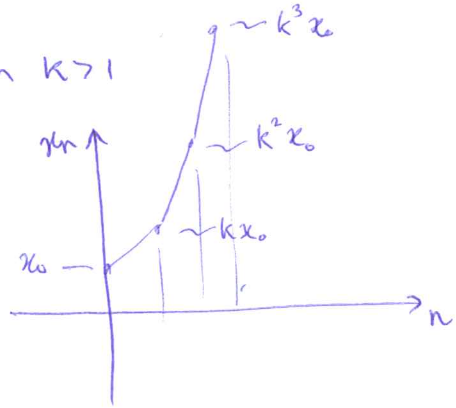
The same equation applies to say bacteria dividing into two every time step ad infinitum. (here x_n is the population of bacterial cells after n divisions). This model assumes that bacteria do not die, of course.

In any case, we might consider the more general equation

$\boxed{x_{n+1} = k x_n}$ — 2

Given some x_0, $\boxed{x_n = k^n x_0}$ — 3

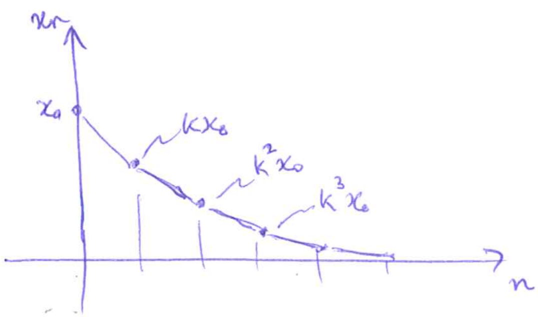
When $k > 1$



$x_n = k^n x_0 \to \infty$ as $n \to \infty$ if $x_0 > 0$.

$\to -\infty$ as $n \to \infty$ if $x_0 < 0$

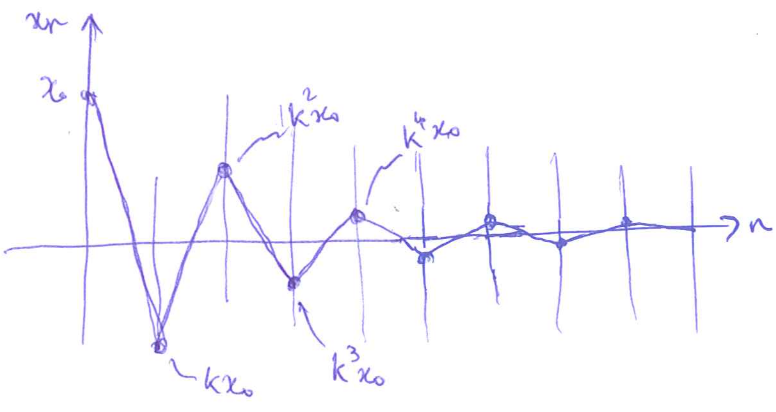
When $0 < k < 1$



$x_n = k^n x_0 \to 0$ as $n \to \infty$ if $x_0 > 0$

$\to 0$ as $n \to \infty$ if $x_0 < 0$.

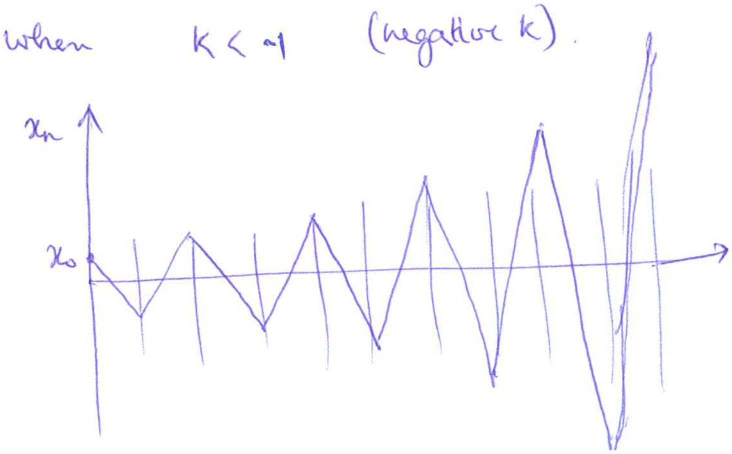
When $-1 < k < 0$ (negative k)



$x_n = k^n x_0 \to 0$ as $n \to \infty$

oscillatory approach to zero.

When $k < -1$ (negative k).



$x_n = k^n x_0$

$|x_n| \to \infty$. as $k \to \infty$.

oscillatory growth to ∞.

(if $x_0 > 0$, $x_n \to \infty$ for n-even

$x_n \to -\infty$ for n-odd.

What if $x_0 = 0$?
$x_n \equiv 0$ for all n (for any k).

$x_{n+1} = k x_n$ has only one fixed point $\underline{x^* = 0}$.

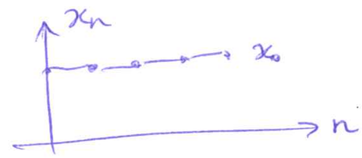
[Fixed point found by setting $x_{n+1} = x_n = x^*$: so solving $x^* = k x^*$ for x^*.].

Fixed point $x^* = 0$

- asymptotically $\boxed{\text{stable}}$ if $-1 < k < 1$, $\boxed{|k| < 1}$.
  (oscillatory approach if $-1 < k < 0$ and monotonic approach if $0 < k < -1$).

- $\boxed{\text{unstable}}$ if $\boxed{|k| > 1}$, i.e., $k > 1$ or $k < -1$.

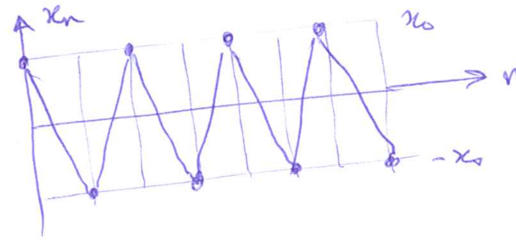
What if $|k| = 1$? (Boundary case)



$k = 1 \Rightarrow x_{n+1} = x_n$. So $x_n = x_0$

Every $x_0 \in \mathbb{R}$ is a fixed point
and they are all neutrally stable.

$k = -1 \Rightarrow x_{n+1} = -x_n$.



x_n oscillates between x_0 and $-x_0$.
"periodic motion". This is called a 2-cycle, because the value of x oscillates between 2 numbers.

$x_{n+1} = k x_n$ is a linear one-dimensional, discrete dynamical system.

Next, we consider the m-dimensional analog.

Linear discrete dynamical system ; m-dimensional case

$$\boxed{Y_{n+1} = A Y_n} \Rightarrow \boxed{Y_n = A^n Y_0}$$ ← analogous to $x_n = k^n x_0$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m.$$ m-vector

Typically, the only fixed point is $Y^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$Y^* = \underset{\sim}{0}$ is asymptotically stable if and only if all eigenvalues of A are real and complex numbers with absolute value strictly less than 1.

Unstable if even one eigenvalue has absolute value greater than 1.

Aside: Finding the FP

Set $Y_{n+1} = Y_n = Y^*$

$\Rightarrow Y^* = AY^*$

$AY^* = Y^*$

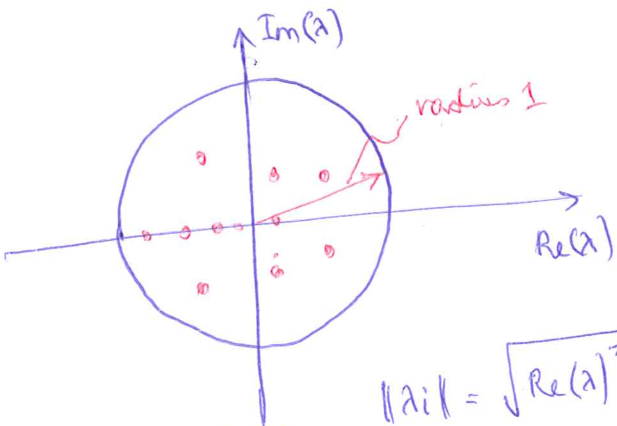
One solution $Y^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

This is the only solution if A has no eigenvalue $= 1$

Otherwise, the line corresponding to the eigenvector of 1 is entirely fixed!

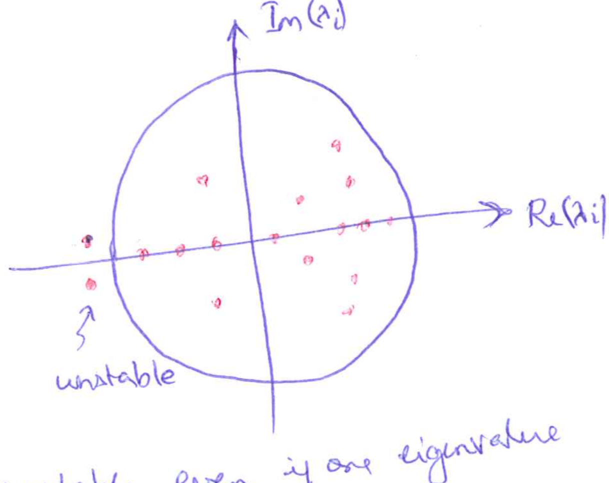
We ignore this for the moment.

Stability condition in pictures. (location of eigenvalues)



$$\|\lambda_i\| = \sqrt{Re(\lambda)^2 + Im(\lambda)^2}$$

asymptotically stable. $\Leftrightarrow$ all eigenvalues within the unit circle. $\|\lambda_i\| < 1$.



unstable even if one eigenvalue is outside the unit circle.

Nonlinear discrete dynamical systems

$$X_{i+1} = f(X_i).$$

some function

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Fixed points are found by solving the equations

$$X^* = f(X^*).$$

what does this equation mean? if $X_i = X^*$, then $X_{i+1} = X^*$ for any i.

For stability, we again find the Jacobian of $f(X)$ with respect to X. at X^*. This is an $N \times N$ matrix.

- Find all the N eigenvalues (by solving, say, the characteristic equation $\det(J - \lambda I) = 0$)
- Sufficient condition for stability $\|\lambda_j\| < 1$ forall i.

  i.e., the absolute value of all the eigenvalues (real or complex) should be less than 1.
- Sufficient condition for instability, one or more of the λ_j have $\|\lambda_j\| > 1$.

Intuition for why the $\|\lambda_i\| < 1$ for stability

$$X_{i+1} = AX_i \quad \text{, say.}$$

Then, $X_1 = AX_0$, $X_2 = AX_1$, etc.

$$\Rightarrow \quad X_i = A^i X_0$$

- Somehow, the matrix A needs to be such that $A^i \to \underline{0}$ as $i \to \infty$ for asymptotic stability.

  zero matrix

- So in a sense the "norm" of A should be less than 1 so that $A^i \to 0$

Which is captured by $\|\lambda_j\| < 1$.

Eg. if A is a diagonal matrix (uncoupled system), with real eigenvalues

then
$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{bmatrix}$$

and
$$A^i = \begin{bmatrix} \lambda_1^i & & & 0 \\ & \lambda_2^i & & \\ & & \lambda_3^i & \\ 0 & & & \ddots \end{bmatrix} \to \begin{bmatrix} & & \\ & 0 & \\ & & \end{bmatrix} \quad \text{as } i \to \infty$$

if and only if $\lambda_j < 1$ for all j.

Justification for why the Jacobian is important in determining the stability of a fixed point

(Linearization near a fixed point)

$X_{i+1} = f(x_i)$. ← some system.

Say X^* is a fixed point of the system, such that $X^* = f(X^*)$

Taylor expand $f(X)$ about X^*.

Near $X = X^*$;

$$f(X) \approx f(X^*) + \underbrace{J(X^*)}_{\substack{\text{Jacobian of} \\ f(X) \text{ evaluated} \\ \text{at } X = X^*}} \cdot (X - X^*) + O(\|X - X^*\|^2).$$

Jacobian of $f(X)$ evaluated at $X = X^*$ – $N \times N$ matrix

So $X_{i+1} = f(x_i)$

$$\approx f(X^*) + J(X^*) \cdot (X_i - X^*) + O(\|X - X^*\|^2).$$

$$\approx X^* + J(X^*) \cdot (X_i - X^*).$$

drop

(Note $f(X^*) = X^*$.)

$$\Rightarrow X_{i+1} - X^* = J(X^*) \cdot (X_i - X^*).$$

Say $\boxed{Z_{i+1} = J Z_i}$, where $Z_i = X_i - X^*$ (vector from X^*).

i.e., the dynamics near the fixed point X^* is governed by the equation $Z_{i+1} = J Z_i$ approximately (at least generically), which is why the Jacobian J is all important.

What if some or all of the eigenvalues are on the unit circle?

Consider $X_{i+1} = A X_i$ linear system.

Example : $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ say. 2D system $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Eigenvalues $A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 + 1 = 0$. $\boxed{\lambda = \pm i}$

- The λ_j both have $\|\lambda_j\| = 1$ $(\lambda = \pm i)$
- Superficially on the boundary of our sufficient condition for asymptotically stable vs unstable.

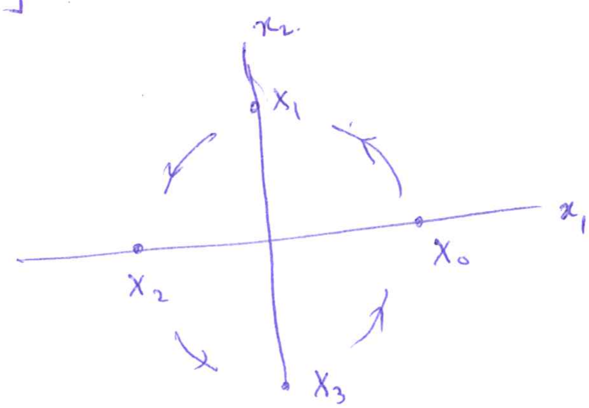
Let us see what this matrix does

Take some initial condition $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$X_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$X_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$X_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



Looks like multiplication by A is a rotation by $90°$. (at least for this initial condition).

Indeed it is. Rotation by some angle θ:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

← general 2D rotation matrix.

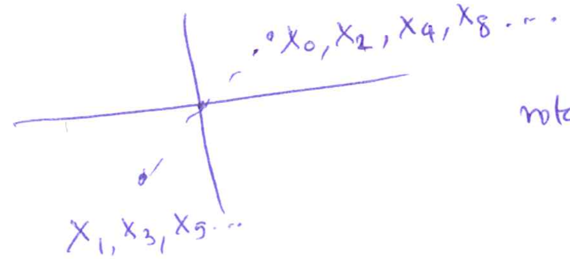
Eigenvalues of this rotation matrix A have all eigenvalues with $\|\lambda_j\| = 1$. In particular, the eigenvalues are

$\cos\theta + i\sin\theta$, $\cos\theta - i\sin\theta$. (show this).

→ When $\theta = 0$, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $x_{i+1} = Ax_i$. Every x^* in the plane is a fixed point.

→ When $\theta = 180°$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. $x^* = 0$ is the only fixed point but every other x_0 goes to $-x_0$ and then comes back to x_0

x_0, $x_1 = -x_0$, $x_2 = -(-x_0) = +x_0$, $x_3 = -x_0$, $x_4 = x_0$, ...



rotation by $180°$ or reflection about origin $\boxed{A^2 x_0 = x_0}$

Every initial condition is a 2-cycle, i.e., repeats every 2 steps.

→ When $\theta = 90°$, rotation by $90°$ $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $x^* = 0$ is the only FP.

Every other point goes through a 4-cycle. $\boxed{A^4 x_0 = x_0}$

→ when θ is an exact fraction of 2π,

say $\theta = \frac{2\pi}{q}$ where q is an integer $\neq 0$

then every initial condition $\neq 0$ is a q-cycle.

That is, $\boxed{A^q X_0 = X_0}$.

→ More generally, when θ is a rational multiple of 2π,

say $\theta = 2\pi \frac{p}{q}$ where $\frac{p}{q}$ is a rational number, p, q integers

with p and q mutually prime (ie, they have no common divisors, so p/q cannot be reduced further).

Then, after q iterations an X_0 has gone through a rotation by $2\pi p$, an integer multiple of 2π, so is back at X_0.

$$A^q X_0 = X_0$$

→ When θ is an <u>irrational</u> multiple of 2π,

ie, $\theta = 2\pi\mu$ where μ is an irrational number

(ie, cannot be represented as p/q, with p, q integers. eg. $\sqrt{2}$) then for $X_0 \neq 0$, X_i is <u>never</u> X_0 for any $i \geq 1$. ie, the net angle X_0 gets rotated by never becomes an integer multiple of 2π.

- In any case, the more important point is that in the linear system $x_{i+1} = Ax_i$, eigenvalues with $\|\lambda_j\| = 1$ correspond to directions / subspaces which never grow or decay asymptotically, but perhaps rotates around in (as in the examples so far).

- For nonlinear systems, one has to look at the other eigenvalues to see if the system has the potential to be stable, and then if appropriate do a nonlinear analysis to check for stability because this is a "boundary" case.

Examples

(1) $x_{i+1} = a x_i (1 - x_i)$.

called the "Logistic map"
This is actually a very famous equation, popularized by Robert May in an article about "CHAOS".

Fixed points

$x^* = a x^* (1 - x^*)$

So $x^* = 0$
(or) $1 = a(1 - x^*)$
$1 - x^* = 1/a$
$x^* = 1 - 1/a$
$= \frac{a-1}{a}$.

2 Fixed points

$f(x) = ax(1-x) = ax - ax^2 = a(x - x^2)$

Jacobian $J = \frac{df}{dx} = a(1 - 2x)$.

$J(x^* = 0) = a$.

$J\left(x^* = \frac{a-1}{a}\right) = a\left[1 - \frac{2(a-1)}{a}\right]$

$= \cancel{a}\left[\frac{a - 2a + 2}{\cancel{a}}\right] = 2 - a$.

$x^* = 0$ locally asymptotically stable if $J = a$

$-1 < a < 1$.

unstable if $|a| > 1$.

$x^* = \frac{a-1}{a}$ $J = 2-a$.

locally asymptotically stable if

$2-a < 1$ & $2-a > -1$

$a > 1$ $a < 3$

$-1 < 2-a < 1 \Rightarrow$

(or) $1 < a < 3$.

unstable otherwise.

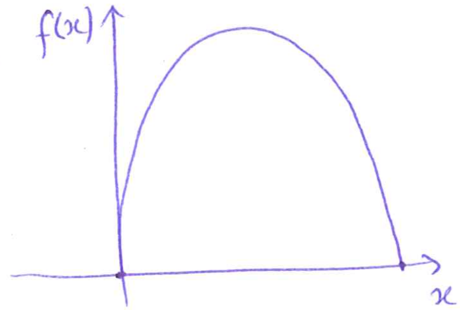
Exercise: Write a computer program to simulate this simple system: $x_{i+1} = a x_i (1-x_i)$.

- Consider different cases of a. $a = 0.5, 1, 2, 3, 4, 5$. (for instance)
- Simulate the system for 2 or 3 initial conditions for each of the different 'a's.
- You will notice that for large enough 'a', for some x_0, the trajectory (x_i) look "chaotic" i.e. at least, superficially, there seems no discernable pattern to the iterates, appears non-periodic (to eye) and definitely does not converge to a fixed point.

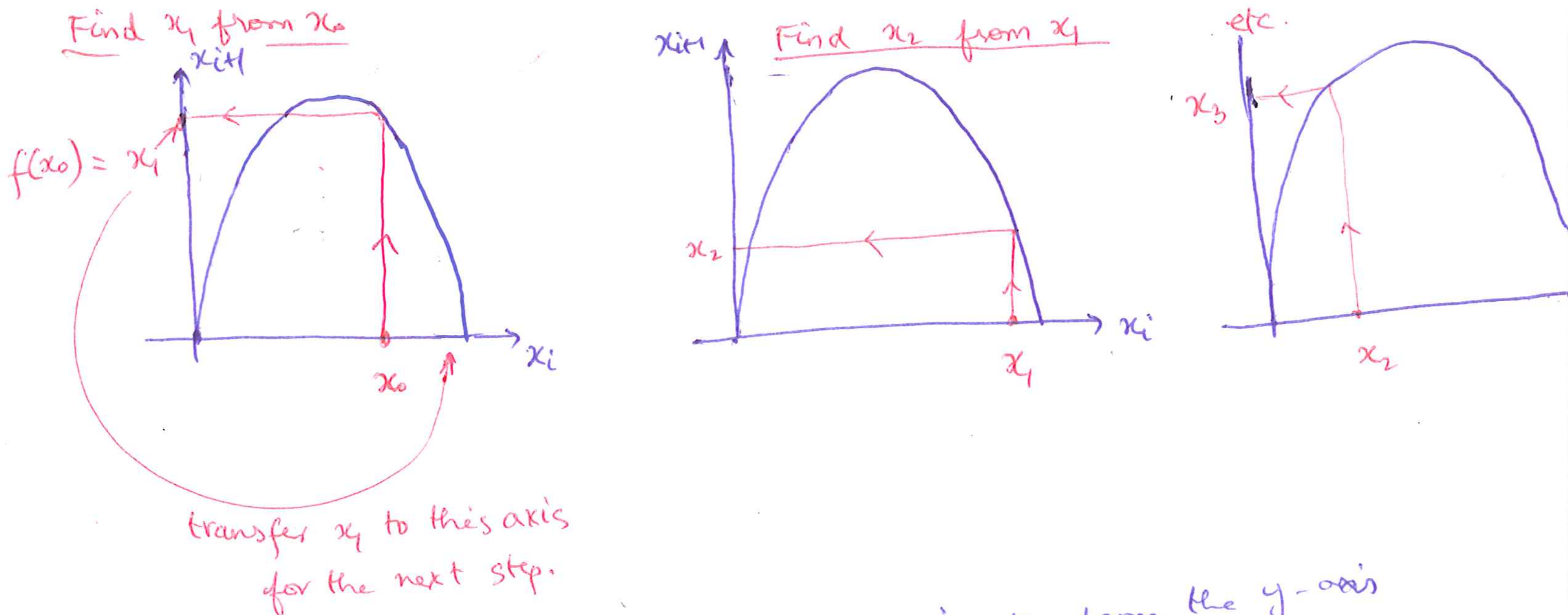
GRAPHICAL TECHNIQUE FOR FINDING ITERATES OF A ONE-DIMENSIONAL MAP (not terribly important in the context of this course).

$x_{i+1} = f(x_i)$. Easiest to consider an example.

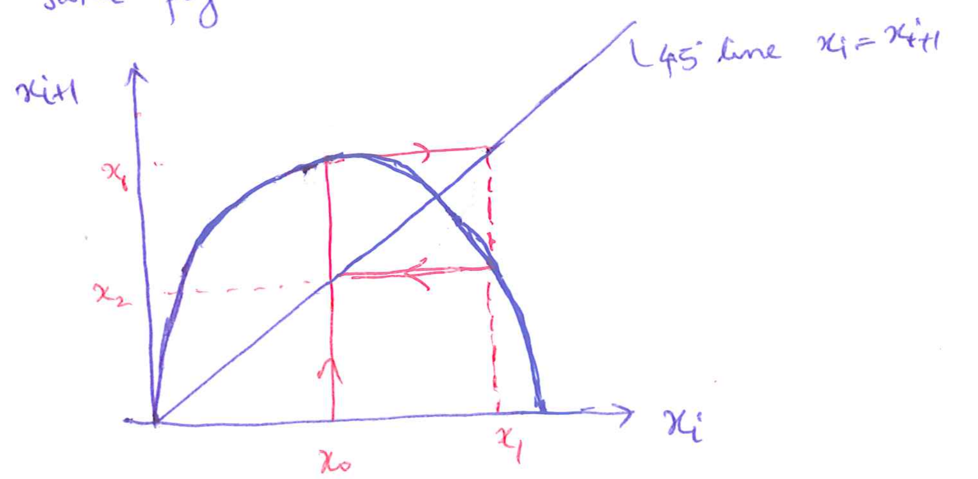
Say $f(x)$ looks like



For example $f(x) = ax(1-x)$.

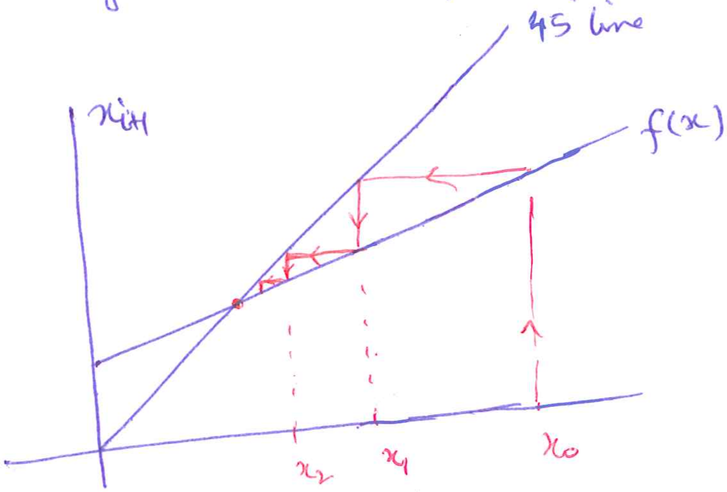


The above method requires transferring the iterates from the y-axis to the x-axis, which can be accomplished by reflection about the 45° line. There is an easier way to do all the iterates in the same figure.



That is, go from x_0 upto $f(x)$. to get x_1. Then go horizontally to the horizontally to the 45° line and then vertically back to the $f(x)$ curve to x_2 and so on.

Example



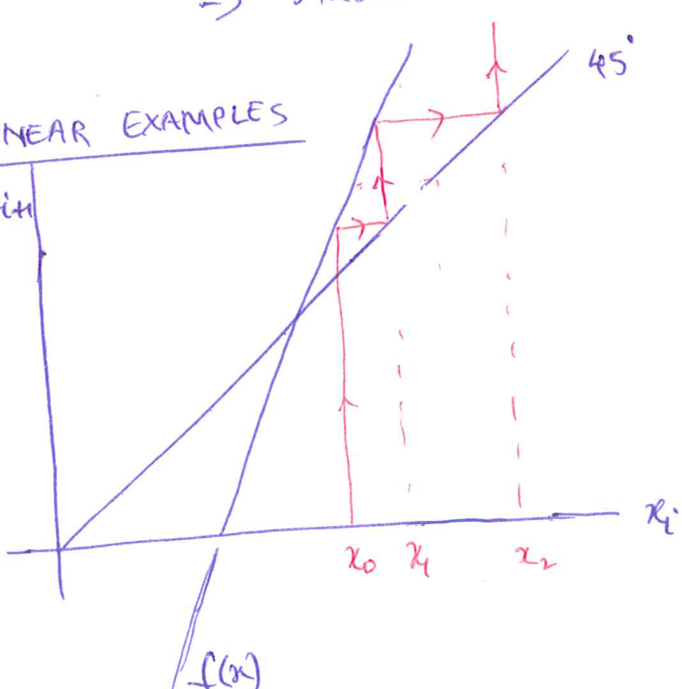
We see that the iterates converge to the point where $f(x)$ intersects the 45° line.

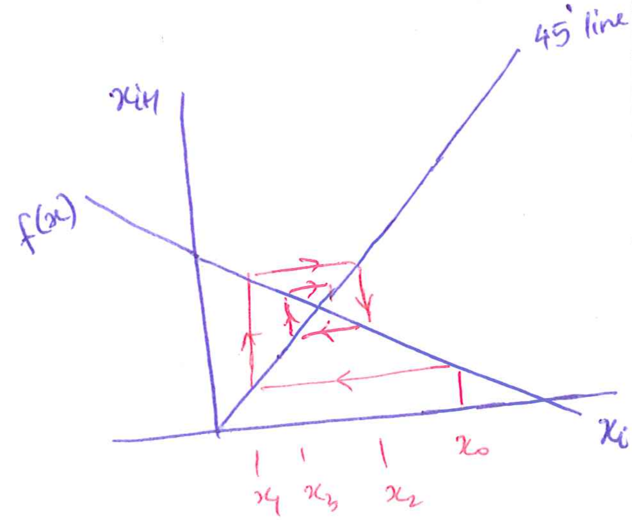
Note 1: Fixed points are where $f(x) = x$ – where $f(x)$ intersects the 45° line.

Note 2: In the above example, we see slope of $f(x) < 1$

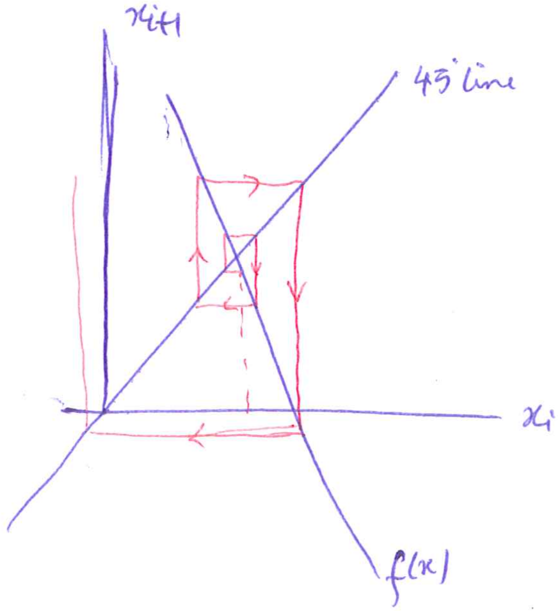
$\Rightarrow$ STABLE

OTHER LINEAR EXAMPLES



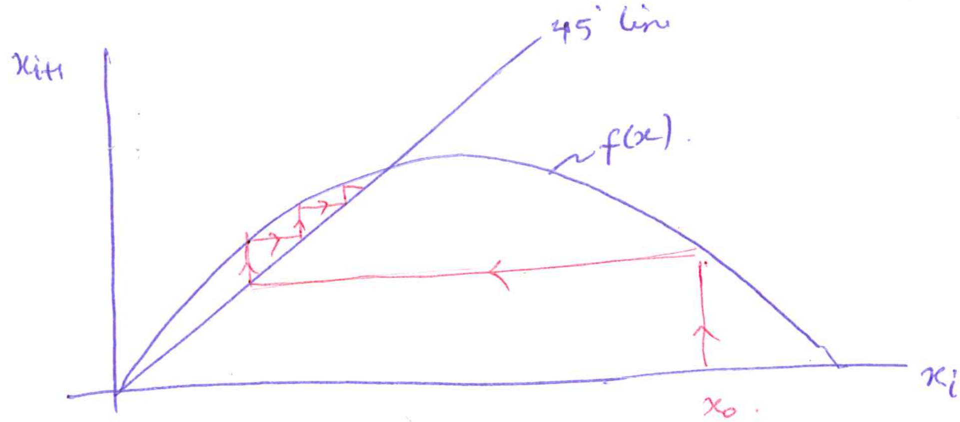


STABLE. with SPIRAL.



UNSTABLE and SPIRAL.

$f(x)$ has negative slope near fixed point.



In any case, these graphical techniques are useful only in one-dimensional systems.