Phase-based models of rhythmic systems

Matthew Kvalheim
Dept. Electrical Engineering and Computer Science,
University of Michigan,
Ann Arbor, United States
Email : kvalheim@umich.edu

Shai Revzen
Dept. Electrical Engineering and Computer Science, Dept. Ecology and Evolutionary Biology,
University of Michigan,
Ann Arbor, United States
Email : shrevzen@umich.edu

I. Introduction

Many biological and engineering systems exhibit periodic behavior. Examples include circadian rhythms, electronic circuits, and heartbeats [1]. A particularly interesting example is that of legged locomotion [2]. In many of these cases, the periodic behavior is *stable*, and so many of these phenomena are well-approximated by nonlinear *oscillator* models – i.e. smooth dynamical systems having an asymptotically stable *limit cycle*, or isolated periodic solution.

In an experimental setting, data is often obtained for a system without *a priori* knowledge of the equations of motion which describe it. If a physical system is well approximated by an oscillator model, one can attempt to deduce the structure of the underlying oscillator model from experimental data from the physical system [3, 4].

In particular, it is often desirable to describe the location and shape of the limit cycle itself. In the biomechanics community, a common practice to estimate the limit cycle representing the "typical stride" of a locomoting animal can be effectively described as follows. First, given time series measurements of an animal's state, data corresponding to distinct strides is partitioned into distinct segments. Next, since the speed of the animal may not be uniform, all of the strides are scaled linearly so that they the same length. This may be thought of as "normalizing time" for each stride. Finally, states corresponding to the same normalized times from distinct strides are averaged. The average corresponding to each normalized time is then assumed to approximate a point on the limit cycle [5].

However, if oscillators are subjected to perturbations, there will be substantial drift in the *phase* of the oscillator [6], leading to uncertainty in the estimation approach described above. In this talk, we will discuss an alternative method of estimating a limit cycle from time-series measurements based on the notion of *asymptotic phase* from dynamical systems theory. Under certain conditions, we provide a bound on the maximum error that our algorithm can produce and prove that, under additional hypotheses, our method provides results superior to the technique described above. Examples will be given from both legged locomotion data sets and simulated oscillator models comparing the performance of these algorithms.

II. Asymptotic Phase-based Estimate

In order to capture the notion of stability, we consider limit cycles which are *orbitally asymptotically stable*; that is, initial conditions in some neighborhood of the limit cycle's path asymptotically approach the cycle [7, 8]. If a dynamical system has a single limit cycle that is both orbitally asymptotically stable and *normally hyperbolic*, we will collectively refer to the stability basin of the limit cycle and the dynamics within as an *oscillator* [3].

Consider the nonlinear oscillator defined by the equation

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathcal{B}^{\text{open}} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^2$ vector field with a single normally hyperbolic, asymptotically stable limit cycle Γ. Let $\phi_t(\cdot)$ be the flow generated by (1). Two points $x, y \in \mathcal{B}$ are said to have the same *asymptotic phase* if

$$\lim_{t\to\infty} \|\phi_t(x) - \phi_t(y)\| = 0$$

The function $\varphi : \mathcal{B} \to S^1$ that assigns to each point in $\mathcal{B}$ its *asymptotic phase* is then a $\mathcal{C}^2$ function of state [1, 9].

Consider the more realistic situation in which (1) is subjected to stochastic perturbations. For simplicity, we consider the case in which the perturbation $b(t)$ is uniformly bounded and is such that a solution to the following ordinary differential equation exists.

$$\dot{x} = f(x) + \delta \cdot b(t) \tag{2}$$

Here $\sup_{t\in\mathbb{R}} \|b(t)\|_2 = 1$ and $\delta > 0$. Let $\gamma(t)$ be the periodic solution of (1) with initial condition $\gamma(0) = x_0$, so that $\gamma(\mathbb{R}) = \Gamma$, and suppose that γ has period T.

Let $X(t; x_0)$ be a solution of (2). We define the *classical estimate* $\hat{c}_t$ of $\gamma(t)$ to be $X(t; x_0)$.

Let $\hat{t} := \inf\{t \in [0, T) : \varphi(X(t; x_0) = t\}$. We define the *phase estimate* $\hat{p}_t$ of $\gamma(t)$ to be $X(\hat{t}; x_0)$

III. Key Results

Proposition 1: Let f, b, φ be as described above. Then there exist constants $L, D > 0$ and a neighborhood U of the limit cycle such that $\forall \delta < D$, for all $x_0 \in U$ and for all $t \in [0, T)$: $\text{var}[\hat{p}_t] \le L\delta^2$

Proposition 2: Let f, b, φ be as described above. Then there exist constants $L, K, D > 0$ and a neighborhood U of the limit cycle such that $\forall \delta < D$, for all $x_0 \in U$ and for all $t \in [0, T)$ such that

$$\int_0^t \int_0^t \nabla\varphi(x(s))\mathbb{E}[b(s)b(\tau)^T]\nabla\varphi(x(\tau))^T ds d\tau > K$$

holds, $\text{var}[X(t; x_0)] > L\delta^2$.

Corollary 1: For t as in the above proposition, the phase estimate of $\gamma(t)$ has smaller variance than the classical estimate.

REFERENCES

[1] A. T. Winfree, *The Geometry of Biological Time*, 2nd ed. New York, New York: Springer-Verlag, 2001.

[2] P. Holmes, R. Full, D. Koditschek, and J. Gukenheimer, “The dynamics of legged locomotion : models, analyses, and challenges,” *SIAM Review*, vol. 48, no. 2, pp. 206–304, 2006.

[3] S. Revzen and J. Guckenheimer, “Finding the dimension of slow dynamics in a rhythmic system,” *J. R. Soc Interface*, 2011.

[4] ——, “Estimating the phase of synchronized oscillators,” *Physical Review E.*, 2008.

[5] M. Daley, A. Voloshina, and A. Biewener, “The role of intrinsic muscle mechanics in the neuromuscular control of stable running in the guinea fowl,” *J. Physiol*, vol. 587, no. 11, pp. 2693–2707, 2009.

[6] A. Demir, A. Mehrotra, and J. Roychowdhury, *Phase noise in oscillators: a unifying theory and numerical methods for characterization*, 2000, vol. 47, no. 5.

[7] J. K. Hale, *Ordinary Differential Equations*, 1st ed. New York, New York: Wiley, 1969.

[8] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, 1st ed. New York, New York: Academic Press, 1974.

[9] J. Guckenheimer, “Isochrons and phaseless sets,” *Journal of Mathematical Biology*, vol. 1, pp. 259–273, 1975.