Symmetry in 3D Walking: Toward Understanding the Natural Dynamics of Legged Locomotion

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I. INTRODUCTION

Achieving stability in 3D legged locomotion is known to be a challenging problem. There have been many different approaches to this problem, ranging from methods based on a linear inverted pendulum approximation of the dynamics to methods that use the full-dimensional dynamic models [2]. However, regardless of the approach, there are two important characteristics in legged locomotion that need to be taken into account: (1) Hybrid Structure; and (2) Underactuation. Without appreciating these two properties, it is hard to understand stability in legged locomotion. We focus on a particular symmetry property that leads to self-synchronization of periodic solutions of some important underactuated hybrid models.

II. SYMMETRIC HYBRID SYSTEMS

A number of low-dimensional models of legged systems, such as the Linear Inverted Pendulum, the Inverted Pendulum, the Inverted Pendulum with Flywheel, and the Spring Loaded Inverted Pendulum are being used as templates for controller design of complex legged systems [1], [2]. Interestingly, all these models share a specific type of symmetry which allows these systems to follow simple *synchronized* periodic orbits. If an appropriate transition map is assumed, all these systems fall into a class of hybrid systems which we refer to as *Symmetric Hybrid Systems*.

Let $q = (x, y)$ be a coordinate system on a 2-dimensional manifold $\mathcal{Q}$ and let $\mathcal{S}$ be a co-dimension 1 submanifold of $\mathcal{TQ}$ defined as the zero set of a function $s : \mathcal{TQ} \to \mathbb{R}$. A $(1, 1)$-dimensional Symmetric Hybrid Control System (SHCS) on $\mathcal{Q}$ denoted by $(f, g, \Delta, \mathcal{S}, \mathcal{U})$ consists of a vector field

$$\ddot{x} = f(x, y, \dot{x}, \dot{y})$$
$$\ddot{y} = g(x, y, \dot{x}, \dot{y})$$

on $\mathcal{TQ}$ satisfying $f(-x, y, \dot{x}, -\dot{y}) = -f(x, y, \dot{x}, \dot{y}), g(-x, y, \dot{x}, -\dot{y}) = g(x, y, \dot{x}, \dot{y})$, and a reset map $\Delta : \mathcal{S} \times \mathcal{U} \to \mathcal{TQ}$ where $\Delta = (\Delta_q, \Delta_{\dot{q}})$ and $\Delta_q(x^-, y^-) = u$ for a control input $u \in \mathcal{U} \subset \mathbb{R}^2$ such that $\Delta_{\dot{q}}(x^-, y^-, \dot{x}^-, \dot{y}^-, u) = (\dot{x}^-, -\dot{y}^-)$ if $u = (-x^-, y^-)$. In

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other words, a legged system modeled by a SHS exhibits odd symmetry in the sagittal plane and even symmetry in the frontal plane.

A SHCS is called an (x_0, y_0)-invariant SHS if
1) $(x_0, y_0, \dot{x}, \dot{y}) \in \mathcal{S}$ for $(\dot{x}, \dot{y}) \in U \times V \subset \mathbb{R} \times \mathbb{R}$ where U and V are open subsets of $\mathbb{R}$.
2) $u = (-x_0, y_0)$

By the definitions above in an (x_0, y_0)-invariant SHS if $x^- = x_0$ and $y^- = y_0$ then $x^+ = -x_0, y^+ = y_0, \dot{x}^+ = \dot{x}^-$ and $\dot{y}^+ = -\dot{y}^-$.

III. EXAMPLES

3D LIP: Figure 1 shows the well-known 3D LIP model. It is the simplest hybrid model with two degrees of underactuation in 3D walking. Its continuous equations of motion are

$$\ddot{x} = \omega^2 x$$
$$\ddot{y} = \omega^2 y$$

which clearly satisfies the odd-even symmetries of a SHS. Defining the switching surface as

$$\mathcal{S} = \{(x, y, \dot{x}, \dot{y}) | x^2 + y^2 = x_0^2 + C y_0^2\} \quad (1)$$

for some $x_0, y_0, C > 0$, and reset map as $\dot{x}^+ = \dot{x}^-, \dot{y}^+ = -\dot{y}^-$ for $(x^-, y^-, \dot{x}^-, \dot{y}^-) \in \mathcal{S}$ completes the model as a SHS. We note that the impact map is lossless.
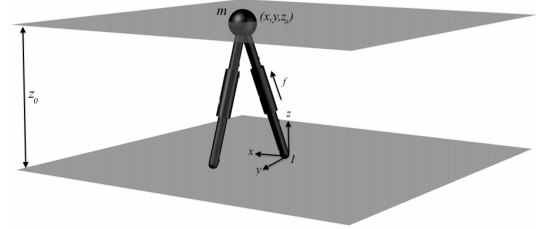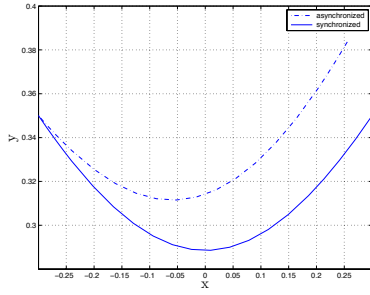

Fig. 1.   3D LIP Biped

Other pendulum models: In a similar manner, the 3D Inverted Pendulum biped, Inverted Pendulum, the Inverted Pendulum with Flywheel, and the Spring Loaded Inverted Pendulum are SHSs if energy losses at impact are neglected by defining an appropriate transition map.

Abstract example: The system

$$\ddot{x} = 10x + (\tan(x) - \sin(x))(\dot{x}^2 + \dot{y}^2) + x e^y + x e^{\dot{x}} + y \dot{y}$$
$$\ddot{y} = 10 y^2 + e^{x^2} + x \dot{y}$$

Fig. 2. A synchronized solution vs. an synchronized solution of a 3D LIP biped model.

Fig. 3. Self-Synchronization. Numbers show the step number.

with a trivial impact map, similar to that of the 3D LIP, is a SHS.

IV. SYNCHRONIZATION AND SELF-SYNCHRONIZATION

A solution $(x(t), y(t))$ of a SHS is said to be *synchronized* if $\exists\, t_m > 0$ such that $x(t_m) = 0$, and $\dot{y} = 0$ whenever $x = 0$. Figure 2 shows a synchronized solution of a 3D LIP model. The importance of synchronized solutions is that they lead to periodic solutions. The odd-even symmetries defined above guarantee the existence of infinitely many synchronized solutions in the form $(x(t), y(t))$ such that $x(t)$ is an odd function of time and $y(t)$ is an even function. Due to this property the restricted Poincaré map of a (x_0, y_0)-invariant SHS, defined on the tangent space of $\mathcal{Q}$ at $(-x_0, y_0)$, has an eigenvalue 1 and another eigenvalue λ. The periodic orbit is said to be self-synchronized if $|\lambda| < 1$. Figure 3 shows an example of a self-synchronized periodic orbit. For the (x_0, y_0)-invariant 3D LIP the eigenvalue 1 corresponds to the neutral stability of kinetic energy and in the case that $C = 1$ in equation (1), the periodic orbit is self-synchronized if $y_0 > x_0$ [6].

V. DISCUSSION TOPICS

Symmetry in Legged Locomotion: In nature, is legged locomotion symmetric in the sense presented above? There have been contradictory views on this question.

Templates and target models: In practice, it has proven remarkably difficult to embed (via feedback and coordinate transformation) the standard pendulum template models into realistic legged models without making approximations [3]. Will it be easier to embed the relaxed conditions associated with SHS?

Decomposition of Stability in 3D: A self-synchronized SHS is neutrally stable in kinetic energy (see the eigenvalue at 1 above). Within an energy level set, asymptotic stability reduces to self-synchronization. To achieve asymptotic stability in the full model, however, asymmetries, such as energy loss at impact and energy gain during single support must be added. This suggests decomposing a legged system as a SHS plus asymmetries, a point of view first proposed by Raibert [4]. In the case of planar walking, where synchronization of sagittal and lateral motions is not an issue, stability concerns kinetic energy.
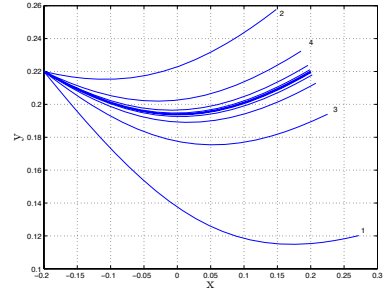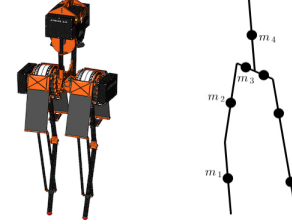

Fig. 4. ATRIAS vs. 9-DOF 3D Biped

VI. CURRENT RESULTS

Current results include an extension of the notion of SHS to higher dimensions (an (m, n)-dimension SHS), determining the structure of the restricted Poincaré map, and preliminary results on introducing asymmetry to stabilize low-dimensional symmetric hybrid systems. Moreover, in the context of high-dimensional biped models, we have examples of determining a set of virtual constraints which result in the zero dynamics being a SHS. This method has been applied to various robot models and stable walking has been achieved. In particular, we obtained stability for the 9-DOF 3D biped with 3 degrees of underacutation shown in Figure 4. This biped is a simplified model of ATRIAS [5].

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