Computing the Finite Time Region of Attraction for Limit Cycles

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1 Introduction

The stability of human and robotic gait has long been an important criteria during the evaluation and construction of locomotor patterns. In contrast to other techniques for comparing gaits (e.g. cost of transport), numerical tools to determine the basin of attraction of a steady state locomotor patterns have been lacking. Given the absence of such tools, various metrics have been proposed to measure the local stability of walking systems such as the Floquet multiplier, the Lyapunov exponent, or the Poincare map. Unfortunately, aside from exhaustive simulation, the determination of the region of attraction of a locomotor pattern remained elusive until the development of Sums-of-Squares Programming which required solving a non-convex optimization problem using bilinear alternations [4].

We present an entirely convex technique using occupation measures to calculate the finite time region of attraction of a steady state locomotion pattern. In addition, we describe how the same technique can be used to simultaneously design a feedback control law that maximizes the size of the finite time basin of attraction of a steady state pattern. While we also use Sums-of-Squares programming, this technique allows the analysis of hybrid polynomial systems up to systems of 8 states. The utility of this approach is illustrated on the rimless wheel and compass gait walkers.

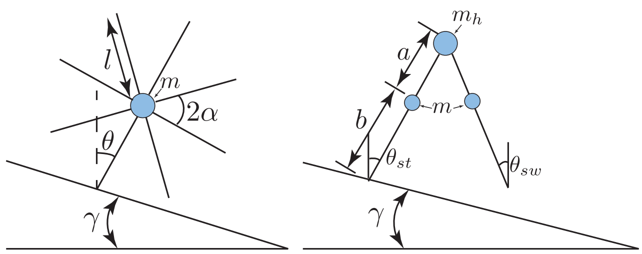
2 Background

To compute the region of attraction to a limit cycle of a hybrid dynamical system, we begin by constructing an efficient description of trajectories at any instant in time using the support of a measure. The relationship between each measure which is defined at distinct times is formalized under the given dynamics via Liouville's Equation [1, 5]. As a result, if a measure at some fixed final time with support equal to the limit cycle is given, then an infinite dimensional linear program can be constructed that is provably able to compute a measure at some initial time whose support is identical to the time limited region of attraction of the given limit cycle.

To implement this infinite dimensional linear program, we use Lasserre's semi-definite hierarchy[2] and assume that the hybrid dynamical system is polynomial. This results in a sequence of convex optimization programs with provably vanishing conservatism each constructing an outer approximation to the true time limited region of attraction of the given limit cycle [5]. In fact, this same approach can be used to perform control synthesis while determining the largest possible time limited region of attraction to a given limit cycle [3].

3 Model

We consider two models of passive biped walking: the rimless wheel and compass gait walking down an inclined slope, shown in Figures 1a and 1b with parameters in Table 1. Using our approach, we compute the finite time region of attraction to the limit cycle for both systems which are illustrated in Figures 2 and 3. For the rimless wheel, we apply our numerical method to the 3rd order Taylor approximation of the dynamics. For the compass gait, we apply our numerical method to the 5th order Taylor approximation of the dynamics about the origin. For both systems, we assume that the measure defined at the final time has support confined to a neighborhood of radius 0.01 around the limit cycle.



(a) Rimless Wheel (b) Compass Gait

Figure 1: Walking Models

4 Results

The computed finite time region of attraction for the rimless wheel walker and the compass gait are shown in Figures 2 and 3 [1], respectively. To evaluate the accuracy of the computed regions, we simulate the trajectories for 5k (and 20k) randomly sampled initial conditions inside (and outside) the

[1]since it has a four dimensional state space, to generate Figure 3, we fix the stance (swing) leg to the limit cycle and find the swing (stance) leg states that are inside the computed region of attraction.

m	l	α	γ	g
1kg	1 m	$\frac{\pi}{8}$	0.2	1 m/s^2

(a) Parameter values for Rimless Wheel

m_h	m	a	b	γ	g
10kg	5kg	$\frac{1}{2}m$	$\frac{1}{2}m$	$\frac{\pi}{60}$	10 m/s^2

(b) Parameter values for Compass Gait

Table 1: Parameters

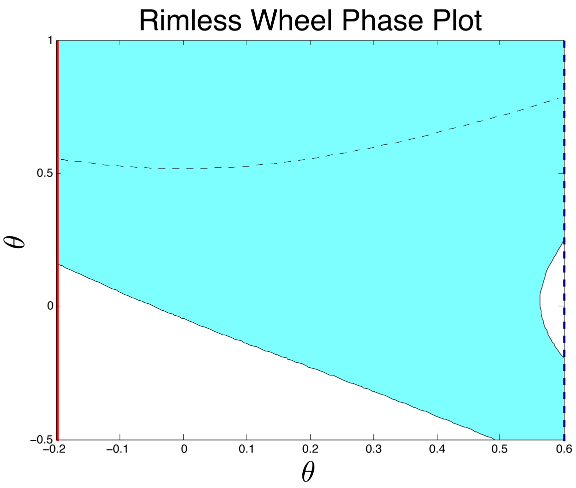


Figure 2: The finite time region of attraction, shaded in blue, for the rimless wheel's limit cycle which is drawn with the dotted black line. The guard is denoted by the dotted blue line on the right and reset to the red line on the left. The time horizon is 20s.

region. For each trajectory, we compute the minimum norm difference to the limit cycle. For the rimless wheel, we consider the percentage of trajectories starting in the region that hit the target set. For the compass gait, we consider the percentage of trajectories that end near the target set, a distance of 0.1 about the limit cycle. The result of this evaluation is described in Table 2.

For the rimless wheel, we find that 96% of initial conditions in the computed region of attraction reaches the target set. For the compass gait, we find that 78% of initial conditions end near the target set.

Model	Inside accuracy (%)	Outside accuracy (%)
RW	96.0	0.0
CG	78.35	2.95

Table 2: Accuracy table for computed region of attraction. Inside accuracy denotes points that start inside the region and reach (end near) the target set. Outside accuracy denotes points that start outside the region and reach (end near) the target set.

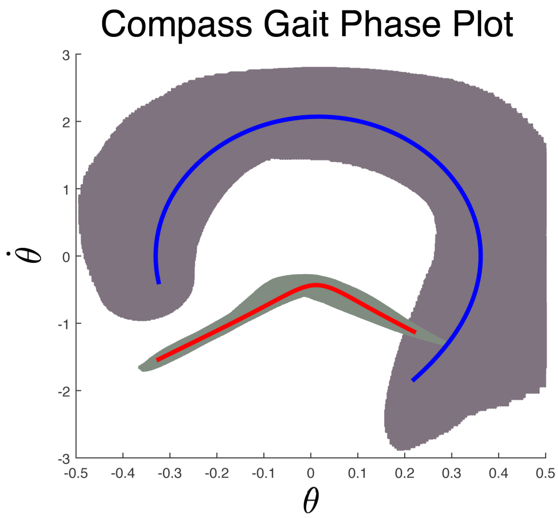


Figure 3: The finite time region of attraction, shaded in gray, for the compass gait's limit cycle. The swing and stance leg limit cycles are projected down to the $(\theta, \dot{\theta})$ domain and denoted by the blue and red line, respectively. The time horizon is 1s.

5 Discussion

We present a convex algorithm to determine the finite time region of attraction for steady state locomotor patterns and illustrate its utility on two common models of gait. Currently, the computed region has an accuracy of 96% for the rimless wheel and 78% for the compass gait. Future work includes incorporating control inputs which will increase the region of attraction of these walking systems and applying this method to additional walking models. Preliminary experiments illustrate that we are able to construct feedback control inputs with the same procedure that more than double the size of the region of attraction to each limit cycle. We are also working on applying this method to higher dimensional models of walking.

References

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