

# Fifteen observations on the structure of energy-minimizing gaits in many simple biped models

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A popular hypothesis regarding legged locomotion is that humans and other large animals walk and run in a manner that minimizes the metabolic energy expenditure for locomotion. Here, using numerical optimization and supporting analytical arguments, I obtain the energy-minimizing gaits of many different simple biped models. I consider bipeds with point-mass bodies and massless legs, with or without a knee, with or without a springy tendon in series with the leg muscle and minimizing one of many different ‘metabolic cost’ models—correlated with muscle work, muscle force raised to some power, the Minetti–Alexander quasi-steady approximation to empirical muscle metabolic rate (from heat and ATPase activity), a new cost function called the ‘generalized work cost’  $C_g$  having some positivity and convexity properties (and includes the Minetti–Alexander cost and the work cost as special cases), and generalizations thereof. For many of these models, walking-like gaits are optimal at low speeds and running-like gaits at higher speeds, so a gait transition is optimal. Minimizing the generalized work cost  $C_g$  appears mostly indistinguishable from minimizing muscle work for all the models. Inverted pendulum walking and impulsive running gaits minimize the work cost, generalized work costs  $C_g$  and a few other costs for the springless bipeds; in particular, a knee-torque-squared cost, appropriate as a simplified model for electric motor power for a kneed robot biped. Many optimal gaits had symmetry properties; for instance, the left stance phase was identical to the right stance phases. Muscle force–velocity relations and legs with masses have predictable qualitative effects, if any, on the optima. For bipeds with compliant tendons, the muscle work-minimizing strategies have close to zero muscle work (isometric muscles), with the springs performing all the leg work. These zero work gaits also minimize the generalized work costs  $C_g$  with substantial additive force or force rate costs, indicating that a running animal’s metabolic cost could be dominated by the cost of producing isometric force, even though performing muscle work is usually expensive. I also catalogue the many differences between the optimal gaits of the various models. These differences contain information that might help us develop models that better predict locomotion data. In particular, for some biologically plausible cost functions, the presence or absence of springs in series with muscles has a large effect on both the coordination strategy and the absolute cost; the absence of springs results in more impulsive (collisional) optimal gaits and the presence of springs leads to more compliant optimal gaits. Most results are obtained for specific speed and stride length combinations close to preferred human behaviour, but limited numerical experiments show that some qualitative results extend to other speed-stride length combinations as well.

**Keywords:** legged locomotion; walking and running; optimization and optimal control; minimize energy; gaits; metabolic cost

## 1. INTRODUCTION

Human walking and running gaits are so stereotypical [1] that many simple, albeit subjective, descriptions of these gaits have been devised by locomotion researchers over the years. Inverted pendulum walking and impulsive running are two such idealized gait descriptions involving impulsive leg work. In inverted pendulum

walking [2–5], the hip moves in a series of circular arcs, vaulting over a straight leg. The transition from one circular arc to the next is accomplished by an impulsive push-off by the trailing leg, followed by an impulsive heel-strike by the leading leg. Other versions of inverted pendulum walking have also been proposed, with various degrees of overlap between heel-strike impulse and push-off impulse [4]. In impulsive running [4–6], the hip describes a series of parabolic free-flight arcs, interrupted by impulsive elastic-looking bounces off a near-vertical leg.

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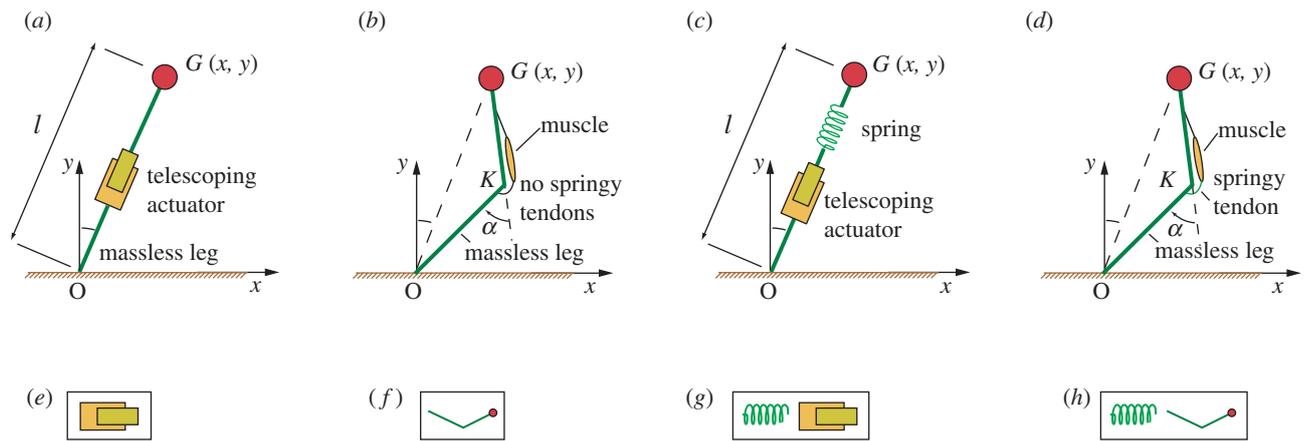


Figure 1. A sequence of simple biped models with different leg architectures. In each biped model, only one leg is shown; the second leg is identical to the one shown. All models have point-mass upper bodies and massless legs. (a) The leg is a telescoping actuator (muscle) with no spring (tendon) in series. (b) The leg has a knee joint, articulated by a muscle without a springy tendon in series. The two segments are of equal length. (c) The leg is a telescoping actuator (muscle) in series with a spring (tendon). (d) The leg has a knee joint, articulated by a muscle in series with a springy tendon. (e–h) Visual abbreviations for each of the four biped models are shown below the corresponding model. These visual abbreviations will be used in other figures to indicate what model those figures refer to.

More recently, the centre of mass motion of human running and walking gaits has been described a little better by the so-called spring-mass models [7–12]. These spring-mass models embody the observation that during walking and running, the leg performs mechanical work more gently than in the ‘impulsive gaits’ described above, undergoing some compression and restitution as if the whole leg were a linear spring.

Why do humans walk and run in the manner approximately described by these simple descriptions? More generally, why do humans and other animals move the way they do? An ancient hypothesis, dating back at least to a contemporary of Galileo and Newton [13], is that animals move in a manner that minimizes effort, perhaps as quantified by metabolic cost per distance travelled [14–16]. Many researchers have attempted to compute the consequences of such energetic optimality for biped models of varying complexity and realism with varying predictive success (e.g. [5, 17–27]). Despite considering relatively different models and assumptions, many of these calculations have resulted in gaits that resemble those of a human in some manner. But it is hard to determine the specific reasons for such agreement as the models are often relatively complex. Also, such model complexity prevents us from appreciating the simple structure of the optimal gaits hiding just beneath the surface, as it were.

Here, using careful numerical optimization and some supporting analytical arguments, I examine biped models with four different leg architectures, minimizing a number of different ‘metabolic cost’ models—correlated with muscle work, muscle force raised to some power, a quasi-steady approximation to empirical metabolic rates, and generalizations and combinations thereof. I find that the optimal gaits for many of these model variations are similar to each other, with the qualitative differences often predictable simply. For instance, I define a class of metabolic cost functions with some positivity and convexity properties called the generalized

work cost  $C_g$ , which includes as a special case both the quasi-steady empirical model and the work cost. Minimizing cost functions from this class of functions gives optimal solutions similar to simply minimizing muscle mechanical work, even though some of these cost functions have large costs for isometric force. Also, I consider biped models that have zero, small or considerable tendon compliance, so the effect of tendon compliance on the optimal gaits can be seen clearly.

A central problem in locomotion biomechanics is the determination of a simple cost function, if it exists, which when minimized for a simple biped model with appropriate task constraints, correctly and quantitatively predicts many aspects of human locomotor behaviour. This paper contributes to the solution of this inverse optimization problem by organizing, to some extent, the consequences of various model assumptions to the optimal gaits. The specific results of the various gait optimizations for the various biped model variations are described in detail in §§7–10. In §12, these results are summarized into a number of observations (about 15 in number) about the structure of energy optimal gaits for these simple biped models. In this section, it is argued that a number of features (gait structure, walk–run transition, various symmetries, walking on straight legs, apparent minimization of work, passive dynamics, linear spring-like leg behaviour, unavailability of joint work, etc.) that characterize human gait might follow from energy minimization. Other significant results and observations (e.g. the ‘optimality of holding still’ and the structure of optimal leg-swing strategies) are noted elsewhere in the paper in appropriate sections and appendices.

## 2. FOUR BIPED MODELS

The four biped models shown in figure 1 are most closely related to those used by Alexander [20, 21] and

Minetti & Alexander [23]. All four biped models have point-mass bodies and massless legs, embodying the observation that the upper body is usually much more massive than the legs (a human leg is about 15% body mass; [28]). The four biped models differ only in the properties of the leg.

The simplest model is shown in figure 1*a*, first introduced by Alexander [20] and elaborated later by Srinivasan & Ruina [5,29] and Srinivasan [30]. In this model, the leg is a telescoping actuator with no elastic spring in series with it; here, the telescoping actuator is identified with the sum total of all muscles articulating all joints, assuming that the muscles have no springy tendons in series.

In the model of figure 1*b*, the massless leg has two segments of equal length connected at a knee, with a uniaxial extensor muscle, without a springy tendon in series. This muscle has a constant moment arm about the joint, so that the joint torque is a constant multiple of the muscle force.

Animals have springy tendons in series with their muscles. These tendons are capable of storing elastic energy when stretched and releasing the energy while shortening with little loss of stored energy, thus potentially lowering the metabolic cost [31,32]. The biped models in figure 1*c,d* attempt to capture this feature by adding a ‘linear’ spring in series with the muscles (actuators) in the previous two models.

The properties of the telescoping actuators and muscles in these biped models are discussed in §5. Figure 1*e–h* show visual abbreviations for each of the four models to be used in other figures in this article.

### 2.1. Equations of motion for the bipeds

All the equations (figures) in this paper are written (drawn) in terms of non-dimensional quantities. All quantities are non-dimensionalized by dividing by appropriate combinations of the maximum leg length  $l_{\max}$ , the body mass  $m$  and the acceleration due to gravity  $g$ . This non-dimensionalization, same as in Srinivasan & Ruina [5] and Srinivasan [30] with slightly different notation, is equivalent to choosing units such that the leg length, the body mass and the acceleration due to gravity are all equal to 1 in their respective units.

The most general non-dimensional equations describing the motion of our biped models are

$$\ddot{x} = F_1(t) \cdot \frac{x - x_{c1}}{l_1} + F_2(t) \cdot \frac{x - x_{c2}}{l_2} \quad (2.1)$$

$$\text{and} \quad \ddot{y} = -1 + F_1(t) \cdot \frac{y}{l_1} + F_2(t) \cdot \frac{y}{l_2},$$

where  $(x, y)$  is the non-dimensionalized position of the point-mass upper body (centre of mass) in the sagittal plane, as shown in figure 1. These equations apply directly to the double support phase, when both feet contact the ground. Here,  $F_1(t)$  and  $F_2(t)$  are the compressive leg forces as functions of time, acting along the lines joining the feet and the upper body,  $x_{c1}$  and  $x_{c2}$  are the  $x$ -positions of the two feet when in ground contact and  $l_1 = \sqrt{(x - x_{c1})^2 + y^2}$  and  $l_2 = \sqrt{(x - x_{c2})^2 + y^2}$  are the lengths of the two legs when they are both in ground

contact. The equations of motion when only one foot touches the ground (single-stance phase) or when neither foot touches the ground (flight phase) are obtained by setting one or both leg forces to zero in equation (2.1).

The non-dimensional stride period is  $T$ , the non-dimensional average speed is  $V = [x(T) - x(0)]/T$  and the non-dimensional stride length is  $D = [x(T) - x(0)]$ . If  $v_{\text{fwd}}$  is the dimensional average speed and  $d_{\text{stride}}$  is the dimensional stride length, then the non-dimensional speed is  $V = v_{\text{fwd}}/\sqrt{gl_{\max}}$  (essentially the Froude number; [33]) and the non-dimensional stride length is  $D = d_{\text{stride}}/l_{\max}$ .

### 3. GAIT OPTIMIZATION PROBLEM STATEMENT IN BRIEF

For each of the four biped models, I seek periodic gaits with at most one stance phase per leg per stride, having a given average speed  $V$ , stride length  $D$  and stride period  $T = D/V$  (all non-dimensional), such that the metabolic cost over the stride period is as small as possible.

The metabolic cost minimized is one of many metabolic cost models discussed in §5. The leg forces and/or muscle forces are constrained either by simple bounds or more elaborate force–velocity bounds. The non-dimensional leg length  $l(t)$  is constrained to be at most 1. The goal is to determine the optimal values for the leg forces as functions of time  $F_1(t)$  and  $F_2(t)$ , appropriate initial conditions, the time periods over which each leg contacts the ground, and the foot–ground contact points.

See the electronic supplementary material for a more detailed formulation of the optimization problems, and the numerical methods used in their solution. The numerical methods are similar to those used earlier by Srinivasan [5,30], with modifications required by the changes in the biped models and cost functions. The electronic supplementary material has brief notes on handling the combinatorial structure of possible bipedal gaits and the regularization of non-smooth functions and kinematic singularities.

Each optimization is performed for a given  $[V, D]$  pair, so ideally one would like to perform gait optimizations for a large number of  $[V, D]$  pairs, thereby determining the dependence of the optimal gait on  $[V, D]$  as in Srinivasan & Ruina [5] and Srinivasan [30]. But because many biped models with many model variations are considered here, it is feasible to explore only a few different  $[V, D]$  pairs in detail. For the gait optimizations described in this paper, I chose  $[V, D]$  pairs mostly on the preferred human speed–stride length relation, given by the simple expression  $D = 2.5V^{0.6}$ , as used by Alexander [20,21]. However, for many of the models, I also explored a few other  $[V, D]$  pairs not close to this preferred human behaviour and briefly allude to the results from these calculations in §12. Other  $[V, D]$  pairs are also implicitly described in the analytical calculations of appendix C.

Finally, it is thought that humans and animals move in a manner that roughly minimizes the total metabolic cost *per unit distance* travelled [14,16,34,35] when not in a hurry. However, when both speed  $V$  and stride length  $D$  are specified, as in the optimization

calculations here, minimizing the metabolic cost per unit time  $\dot{E}_m$ , the metabolic cost per unit distance  $\dot{E}_m/V$ , and the metabolic cost per stride  $\dot{E}_m \cdot D/V$  are all mathematically equivalent, and will necessarily give the same optimal gaits (see [16] for a discussion).

#### 4. TOTAL METABOLIC COST OF LOCOMOTION

For simplicity, I model the total metabolic cost for locomotion as being separable into three additive parts: (i) resting cost, (ii) stance cost, and (iii) leg-swing cost [2,29,36].

**Resting cost.** The resting cost  $\dot{E}_r$  is a constant equal to the nominal metabolic rate at rest. For most of this paper, the objective is to determine gaits that minimize the energetic cost per unit distance travelled at specified average speeds  $V$  and stride lengths  $D$ . When both the speed and stride length are specified, the resting cost per unit distance ( $\dot{E}_r/V$  speed) only contributes a constant to the total cost per distance, and, therefore, drops out of the optimization. An aside: in many models without a resting cost, the total cost per distance is a strictly increasing function of the forward speed, making the optimal walking speed zero; the resting cost makes the optimal speed non-zero [37,38], but as described above, is not relevant when the speed is specified. See [16] for an account of issues related to optimal walking and running speeds.

**Stance cost.** The stance cost over a stride is the metabolic expenditure of the legs, either the telescoping actuator or the knee muscle in figure 1, as the legs exert force and perform work on the upper body while in contact with the ground (stance phase). This stance cost depends in detail on the motion of the centre of mass. For almost all of this paper, the goal is to determine gaits that minimize the stance cost for a given speed and stride length.

**Swing cost.** The possible cost of swinging the legs is neglected, as is consistent with massless legs. I constrain both the speed and the stride length in the optimizations because it is usually thought that a leg-swing cost is important for determining the optimal stride length when given only the forward speed [30,36]. The effect of adding legs with masses, with an attendant cost for leg swing [28], is briefly discussed in §11. As noted in Srinivasan & Ruina [5], if the leg-swing cost is a pure function of  $V$ ,  $D$  and  $T$  (which it is not quite), it drops out of the optimization at any specified  $[V, D]$ .

#### 5. MUSCLE MODELS

##### 5.1. Metabolic cost models and force-velocity relations

One might expect the energy optimal gaits to depend somewhat on the specific properties of the muscles and actuators. A muscle model specifies how the muscle's length  $l_m$ , shortening rate  $v_m$ , activation  $a$  ( $0 \leq a \leq 1$ ) and possibly other relevant muscle state variables (like the strain distribution of the bound cross-bridge population) affect the tensional muscle

force  $F_m$  ( $\geq 0$ ) and the metabolic energy expenditure of the muscle.

**Work cost  $C_w$ .** The simplest muscle model considered here expends energy at a rate proportional to a weighted sum of the positive and negative mechanical power produced by it (as in [5,20]):

$$\text{metabolic cost, } C_w = \int (b_1[P]^+ + b_2[P]^-) dt, \quad (5.1)$$

where  $P = F_m v_m$  is the mechanical power of the muscle, the positive part  $[P]^+$  of the power is defined as:  $[P]^+ = P$  when  $P \geq 0$  and  $[P]^+ = 0$  when  $P < 0$ . The integral of  $[P]^- (= [-P]^+)$  gives the negative work. This cost model is motivated by the observation that a muscle's metabolic rate is somewhat correlated with mechanical power, but with different positive constants of proportionality  $b_1$  and  $b_2$  (inverse efficiencies) for positive and negative work, respectively [4,39]. The primary purpose of this simple model here is to serve as a reference for what 'minimizing work' entails, and will often be called the 'work cost'. Note that isometric force generation (force without work) has no cost in this model and negative work cannot be stored in the muscle to perform positive work later.

**Generalized work cost  $C_g$ .** It is useful to define a (new) more general cost function  $C_g$  of the form:

$$\text{metabolic cost } C_g = \int F_m \cdot g(v_m) dt, \quad (5.2)$$

where  $g(v_m)$  is some positive and convex function of the muscle-shortening velocity, and the muscle force obeys  $F_m \geq 0$ . The work cost  $C_w$  (equation (5.1)) is a special case of  $C_g$  with  $g(v_m) = b_1[v_m]^+ + b_2[v_m]^-$ . Another simple special case is a linear combination of a work cost and the integral of muscle force, suggested by Alexander [40], with  $g(v_m) = b_1[v_m]^+ + b_2[v_m]^- + b_3$ . The function  $g(v_m)$  is the 'cost per unit force magnitude' at the shortening rate  $v_m$ .  $C_g$  is called the generalized work cost because minimizing  $C_g$  often gives optimal behaviour similar to minimizing  $C_w$ , even though it can have substantial costs for isometric force.

**Convexity and positivity** of  $g(v_m)$  will be key determinants of the qualitative structure of the optimal gaits. In §6, I elaborate on what one means by convexity and show that convexity is necessary to ensure meaningful optimal solutions in a simple but fundamental example of 'supporting a weight'. In appendix B, I discuss another important task that suggests positivity as a plausible property. In this paper, I will only use  $C_g$  with positive and convex  $g(v_m)$  in the optimizations. The  $g(v_m)$  for  $C_w$  is piecewise linear and (barely) convex.

**Alexander-Minetti quasi-steady empirical cost  $C_{AM}$ .** Next, I consider a simplified Hill-type model, borrowed from Alexander and Minetti [23,41,42]. In this model, the muscle force depends on muscle activation  $a$  and shortening velocity  $v_m$ , but not directly on muscle length  $l_m$ . In the following,  $v_{max}$  is the maximum shortening velocity at which a positive muscle force can be produced, and  $F_0$  is the isometric muscle force when fully activated ( $v_m = 0$ ,  $a = 1$ ). Figure 2*a,b* shows the normalized force

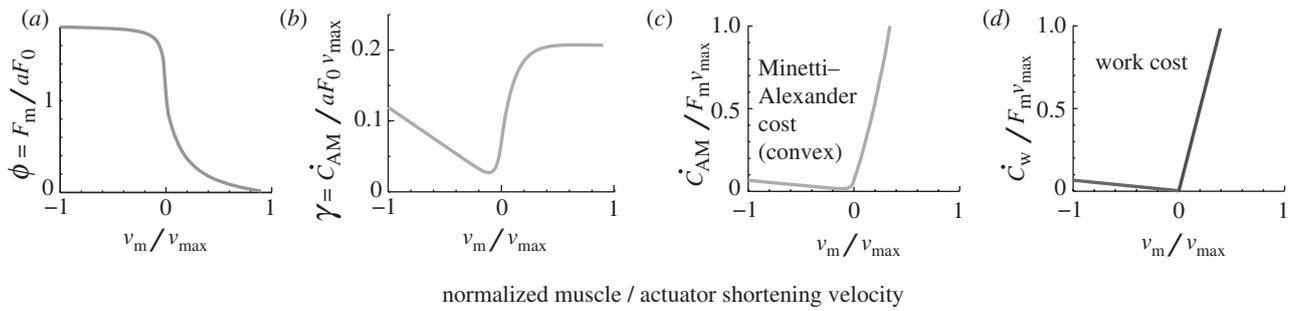


Figure 2. A quasi-steady muscle model. Dependence of (a) the muscle force  $F_m$  and (b) the muscle metabolic rate  $\dot{C}_{AM}$  on the activation  $a$  and normalized shortening velocity  $v_m/v_{max}$ , given the isometric force  $F_0$  and the maximum shortening velocity  $v_{max}$  at which the muscle can produce force. (c) The Alexander–Minetti empirical metabolic rate per unit muscle force based on heat and ATPase activity  $\dot{C}_{AM}/F_m v_{max} = g(v_m/v_{max})$ . (d) The work cost per unit force  $\dot{C}_w/F_m v_{max} = (b_1[v_m]^+ + b_2[v_m]^-)/v_{max}$ . The  $b_1$  and  $b_2$  for (d) were selected so that it most closely resembled (c).

$F_m/aF_0 = \phi(v_m/v_{max})$  and the normalized empirical cost rate  $\dot{C}_{AM}/aF_0 v_{max} = \gamma(v_m/v_{max})$  as a function of the normalized muscle-shortening velocity  $v_m/v_{max}$ . This is a quasi-steady muscle metabolic cost model, an approximation to some roughly steady-state constant- $v_m$  experimental data on muscle heat generation and ATPase activity at full activation (reviewed in [43,44]), with an extra assumption of linear dependence of both the force and the cost rate on the muscle activation.

Observe that  $C_{AM}$  can also be written in the form  $C_g$  (equation (5.2)). The  $g(v_m) = \dot{C}_{AM}/F_m = v_{max}\gamma/\phi$  for  $C_{AM}$  is depicted in figure 2c. A  $g(v_m)$  for  $C_w$  with appropriately chosen  $b_1$  and  $b_2$  is shown in figure 2d to emphasize the similarity of Alexander–Minetti empirical metabolic cost  $C_{AM}$  and the work cost  $C_w$ . This apparent similarity between the cost  $C_{AM}$  and the work cost  $C_w$ , and the specific plots (figure 2c,d) were first suggested to me by A. Ruina (2008, personal communication). The  $g(v_m)$  for  $C_{AM}$  is convex or close-to-but-not-quite convex depending on the approximation of the experimental data.

**Muscle force costs and robot electric motor costs.** Another cost function commonly used in the gait optimization literature is:

$$C_{F\beta} = \int |F_m|^\beta dt. \quad (5.3)$$

I mainly consider  $\beta = 2$ , giving  $C_{F2}$ . The  $\beta = 1$  case is an exceptional case of  $C_g$ , with  $g(v) = \text{constant}$  (see §6 for comment). The  $C_{F2}$  cost or a related activation-squared cost has been used by numerous authors as a proxy for a (metabolic) cost function, especially in the context of inverse dynamics, when one wishes to compute the individual muscle forces consistent with an observed body motion when there are many more muscles than degrees of freedom in the model [24,45,46]. While this procedure often gives reasonable agreement with muscle EMG data, there is no direct biophysical basis for such a cost depending purely on the muscle force; it contradicts the so-called Fenn effect [47] and (equivalently) the muscle data that motivate the cost  $C_{AM}$  just described. When the force is replaced by a torque,  $C_{F2}$  is also perhaps the simplest

cost model for the energy cost for electric motors and therefore relevant for legged robots<sup>1</sup> [48–51].

**Cost combinations.** The metabolic cost functions described so far (equations (5.1)–(5.3) and figure 2b) are only simple models. So I also consider convex combinations of some cost functions: for instance,  $C = \lambda C_w + (1 - \lambda)C_{F2}$ , where  $0 \leq \lambda \leq 1$ . Such cost combinations are not meant as actual models for metabolic costs, but are considered mainly to examine the robustness of the optimal solutions to perturbations in the costs.

**Force–velocity relations.** Among the numerous optimizations performed for this paper, only a few involved a physiological force–velocity relation, such as that shown in figure 2a. For the rest of the optimizations, only a simple muscle force bound is used. Sometimes, a linear force–velocity relationship, such as  $F_m \leq F_0(1 - v_m/v_{max})$ , completely specified by  $F_0$  and  $v_{max}$ , was used for simplicity of computation.

## 5.2. Muscle forces and length rates

The above metabolic cost functions and the force–velocity relations are expressed in terms of the muscle force  $F_m$  and the muscle-shortening rate  $v_m$ . Because of the differences in body architectures, I now describe what specifically is meant by  $F_m$  and  $v_m$  in each of the four models.

**Springless telescoping biped.** The muscle is identical to the telescoping actuator, so the muscle force is identical to the leg force  $F$ . The muscle-shortening velocity is equal to the leg lengthening rate ( $v_m = \dot{l}$ ).

**Springless kneed biped.** The muscle force is proportional to the knee moment and the muscle length shortening rate is proportional to  $-\dot{\alpha}$ , the negative of the knee angle rate. The knee moment  $M$  is given by  $M = -(F/2) \cdot \sin(\alpha/2)$ , where  $F$  is the leg force and  $\alpha$  is the knee angle. The knee angle by  $l = \cos(\alpha/2)$ . The knee angular velocity  $\dot{\alpha}$  is given by:  $\dot{l} = -\dot{\alpha}$

<sup>1</sup>Actually, the simplest model for electric motors would be  $a_1 F^2 + a_2 P$ , the first term related to Joule heating losses and the second term allowing for regeneration of mechanical power. The power  $P$  gets integrated to zero when one considers periodic motions in the absence of physical dissipation, as is the case here, so the second term drops out of the optimizations here.

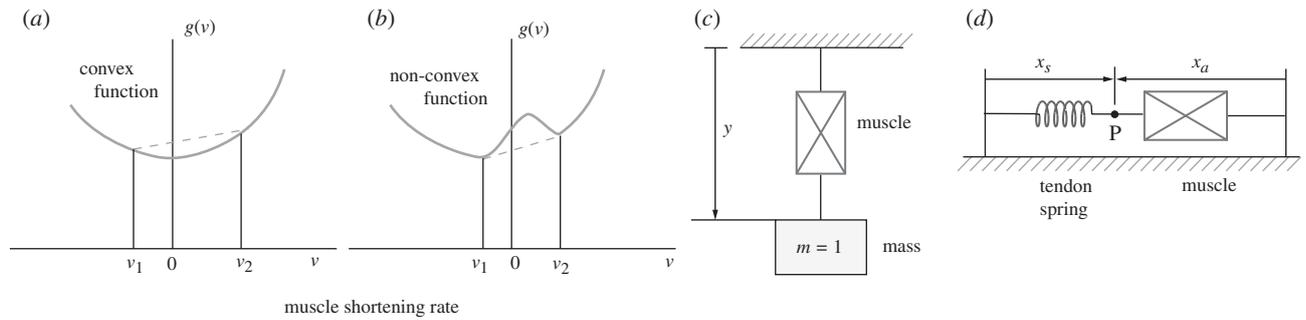


Figure 3. (a) A convex function. For the convex function, the function value (solid line) at some velocity  $v$  is less than or equal to any weighted average (dotted line) of the function values at two or more velocities (say  $v_1$  and  $v_2$ ) whose weighted average is  $v$ . (b) A non-convex function. For the non-convex function, there exist velocities  $v_1$  and  $v_2$  such that the aforementioned inequality is violated. (c) The objective is to keep the mass at a given position on average, with zero average velocity. (d) The objective is to exert a given average force on the wall, with possibly some periodic motion of the point P.

$\sin(\alpha/2)/2$ . The knee muscle force is  $F_m = |M|/r_m$ , where  $r_m$  is the constant moment arm of the muscle about the knee joint. Similarly, the knee muscle-shortening rate is  $v_m = -r_m\dot{\alpha}$ . The actual value of the moment arm  $r_m$  plays no role in this paper, because all the calculations—the evaluations of the metabolic cost, the dynamics and the constraints—are expressed directly in terms of the knee moment  $M$  and the knee angular velocity  $\dot{\alpha}$ .

**Springy telescoping biped.** The muscle force is identical to the leg force  $F$  here. The total leg length  $l$  is the sum of two lengths: tendon spring's  $l_{\text{spring}}$  and the telescoping actuator's  $l_{\text{actuator}}$ . When the spring is linear with non-dimensional stiffness  $k$ , the leg force is equal to  $F = -k(l_{\text{spring}} - l_0)$ , where  $l_0$  is the spring's stress-free length. The spring length rate is then  $\dot{l}_{\text{spring}} = -\dot{F}/k$ . The actuator length rate is  $\dot{l}_{\text{actuator}} = \dot{l} - \dot{l}_{\text{spring}}$ . The muscle-shortening rate is  $v_m = \dot{l}_{\text{actuator}}$ . In this paper, the quantities  $l_{\text{actuator}}$  and  $l_{\text{spring}}$  appear in calculations either as their derivatives or their sum  $l$ , so that their individual values are irrelevant and left undetermined up to a constant.

**Springy kneed biped.** The muscle force is proportional to the knee moment as before ( $F_m = |M|/r_m$ ). The knee flexion angle  $\alpha$  is accommodated by changes in the spring length and the muscle length, which are quantified in terms of the knee angle changes they are responsible for:  $\alpha_{\text{spring}}$  and  $\alpha_{\text{muscle}}$ . The spring deflection is related to the knee moment by  $M = -k\alpha_{\text{spring}} = -(F/2) \cdot \sin(\alpha/2)$ , assuming a torsional spring stiffness  $k$ . The spring angular rate is given by  $\dot{\alpha}_{\text{spring}} = -M/k = (\dot{F}/2k) \sin(\alpha/2) + (F\dot{\alpha}/4k) \cos(\alpha/2)$ . The muscle angular rate is then  $\dot{\alpha}_{\text{muscle}} = \dot{\alpha} - \dot{\alpha}_{\text{spring}}$ . The muscle length shortening rate is  $v_m = -r_m\dot{\alpha}_{\text{muscle}}$ . Again, the moment arm  $r_m$  of the muscle drops out of all calculations here.

## 6. 'HOLDING STILL' IS OPTIMAL FOR $C_G$ IF $G(V)$ IS CONVEX

In §5, a class of candidate muscle cost functions called the generalized work cost,  $C_g = \int Fg(v) dt$ , was introduced. Before this class of cost functions is used in the locomotion optimization problems, I use this cost function in the context of the much simpler task of 'exerting

a given average force' using a muscle, with the least cost. This problem is described to show why  $g(v)$  must be *strictly convex* around  $v=0$  and what happens if  $g(v)$  were *not convex*. This section is perhaps slightly digressive (but fundamental), and the reader may skip to the next section without much loss of continuity.

**Supporting an average force.** Consider the situation shown in figure 3c. The goal is to use the muscle to support the mass ( $m=1$ ) at some vertical position on average, in the presence of gravity, while minimizing the cost  $C_g$  with a strictly convex  $g(v)$ , over some long time period (or per unit time). I wish to show that holding the mass exactly at rest for all time (that is, holding still) is less expensive than, say, moving the mass about an average position periodically with any period however long.

So consider, as an alternative, a time-varying periodic force  $F(t)$ , producing a time-varying vertical velocity  $v = -\dot{y}$ . Say the time period of this oscillation is  $t_p$ . The equation of motion is  $\dot{v} = F - 1$  (gravity constant = 1). The periodicity equations imply

$$\begin{aligned} v(0) = v(t_p) &\Rightarrow \int_0^{t_p} (F - 1) dt = 0; \\ &\Rightarrow \int_0^{t_p} \frac{F(t)}{t_p} dt = 1 \end{aligned} \quad (6.1)$$

and

$$y(0) = y(t_p) \Rightarrow \int_0^{t_p} v(t) dt = 0. \quad (6.2)$$

**Definition: strictly convex function.** A function  $g(v)$  is said to be strictly convex if for every  $v_1 \neq v_2$  in its domain, it satisfies

$$g(\mu v_1 + (1 - \mu)v_2) < \mu g(v_1) + (1 - \mu)g(v_2), \quad (6.3)$$

where  $0 < \mu < 1$ . In words, the 'function value at a weighted average of two  $v$ 's' is less than the 'weighted average of the function values at two  $v$ 's'. More colloquially, convex functions are functions that are 'bowl shaped'. Figure 3a,b shows examples of convex and non-convex functions. In figure 3a,b, if some point on the solid line represents the left-hand side of equation (6.3), the point on the dotted straight line with the

same abscissa ( $v$ -value) represents the right-hand side of equation (6.3).

**Proof of optimality.** Strictly convex functions obey the so-called Jensen's inequality [52]:

$$g\left(\int_0^{t_p} \mu(t)v(t) dt\right) < \int_0^{t_p} \mu(t)g(v(t)) dt, \quad (6.4)$$

where  $\mu(t)$  is any non-constant non-negative (piecewise continuous) function that satisfies  $\int_0^{t_p} \mu(t) dt = 1$  and  $v(t)$  is non-constant. Equation (6.4) is essentially a generalization of equation (6.3) in the sense that equation (6.4) involves weighted averages over a possibly continuous distribution of  $v$ , as opposed to averages over only two distinct  $v$ 's in equation (6.3). Setting  $\mu(t) = F(t)/t_p$  in equation (6.3),

$$g\left(\int_0^{t_p} \frac{F(t)}{t_p} \cdot v(t) dt\right) < \int_0^{t_p} \frac{F(t)}{t_p} \cdot g(v(t)) dt = \frac{C_g}{t_p} \quad (6.5)$$

for non-constant  $F(t)$ . The right-most equality with  $C_g/t_p$  above assumes that  $F(t)$  is positive. Also,  $\int_0^{t_p} F(t)v(t) dt = 0$  because this is the mechanical work done over one period and integrates to zero by periodicity. So, equation (6.5) simplifies to

$$g(0) < \frac{C_g}{t_p} \quad \text{or} \quad g(0)t_p < C_g. \quad (6.6)$$

That is, the cost  $C_g$  with any non-constant periodic  $F(t)$  over time  $t_p$  will always be greater than the  $g(0)t_p$ , which can only be attained by keeping the mass absolutely still:  $v(t) \equiv 0$ . Thus, 'holding still' is the unique optimal strategy (at least from among periodic motions of arbitrarily long periods) if  $g(v)$  is strictly convex.

**Effect of non-convexity.** If  $g(v)$  were not convex (concave) around  $v=0$ , for instance, as in figure 3*b*, there exists a time-varying  $\mu(t)$ , and therefore  $F(t)$ , for which the above inequalities are reversed; that is,  $C_g \leq g(0)t_p$  and a time-varying  $F(t)$  ('tremor') would be better than holding still. In the non-convex example of figure 3*b*, one would be able to reduce the effective  $g(v)$  to the dotted line, from the solid line, by operating between  $v_1$  and  $v_2$ . It is not clear if such optimality of oscillation, as opposed to rest, is physiological. A provocative conjecture would be that muscle activation is slightly 'noisy' to take energetic advantage of the possible slight non-convexity in the cost function, a hypothesis I hope to examine elsewhere in greater detail. To keep things simple,  $g(v)$  is restricted to convex functions here.

Note that strict convexity of  $g(v)$  is sufficient but not necessary for the above result. Optimality of holding still only requires that the inequality (equation (6.3)) be strict when the weighted average of  $v$  evaluates to  $v=0$ . For instance, functions like  $g(v) = |v|$ , while not strictly convex away from  $v=0$ , do have holding to be optimal for this task. Most significantly, note also that the optimality of holding still does not depend at all on the minimum of  $g(v)$  being at  $v=0$ ; for instance, even if  $g(v) = 1 + (v-1)^2$ , holding still ( $v(t) \equiv 0$ ) is optimal. Nor is it necessary that  $g(v)$  be positive.

**Another situation.** The optimality of holding still also applies to other situations; for instance, that shown in figure 3*d*, in which it is required that the muscle exert an average force of unit magnitude on the wall, as it perhaps oscillates against the spring. Set  $v = \dot{x}_a$ . By periodicity,  $x_a(0) = x_a(t_p)$  and  $x_s(0) = x_s(t_p)$ , giving  $\int_0^{t_p} F(t)v(t) dt = 0$  (no net work done). Jensen's inequality (equation (6.5)) again applies if  $g(v)$  is strictly convex, giving  $g(0)t_p < C_g$ , demonstrating the optimality of holding still.

**Positivity.** In appendix B, a simple task analogous to swinging a leg is considered. In this leg swing task, positivity of  $g(v)$  is required to avoid some implausible optimal strategies. Positivity of  $g(v)$  is natural; it simply means that the metabolic rate is positive whether the muscle is performing positive or negative work. Thus, in this paper,  $g(v)$  is both positive and convex.

## 7. SPRINGLESS TELESCOPING BIPED

As the simplest in the series of four models (figure 1*a-d*), I discuss the optimal gaits of the springless telescoping biped (figure 1*a*) in greater detail than that of the other models. Srinivasan and Ruina [5,30] considered the simplest version of this model, determining the periodic gaits that minimized the work cost  $C_w$  at a range of speeds and step lengths  $[V, D]$ , assuming that the leg forces can be unbounded if necessary. They discovered that inverted pendulum walking was optimal at low speeds, impulsive running was optimal at higher speeds and a third hybrid gait dubbed 'pendular running' was optimal at a range of intermediate speeds and large step lengths. However, this calculation did not allow double support, considered only a single step, assumed that each step is the same as every other step and did not minimize multiple cost models other than  $C_w$ . In the following, I rid the calculations of these simplifications. I assume bounded leg forces, allow for double-support phases, consider two steps (equal to one stride) instead of one step, with the possibility that the steps might be different, and consider multiple muscle metabolic cost models. In the rest of the paper, unless specified otherwise, the maximum possible leg force is 3 (i.e. three times body weight). This choice for the maximum leg force has no special significance except being sufficiently greater than 1 to support many gaits, but not unrealistically high. Values such as 2.5 or 3.5 give qualitatively the same answer for a range of  $[V, D]$ .

**Minimize work.** Minimizing  $C_w$  with these (mostly) relaxed constraints does not produce any gaits qualitatively different from those found in Srinivasan & Ruina [5]. Mostly, I found that non-impulsive analogues of the three impulsive gaits—namely, inverted pendular walking, impulsive running and pendular running [5]—minimized  $C_w$ . That is, the infinite forces required during the impulses of the three previously optimal impulsive gaits are replaced by the maximum available leg forces (here, equal to 3) in these non-impulsive gaits.

For example, figure 4*a-c* shows the non-impulsive analogue of inverted pendular walking, optimal when  $V = 0.39$  and  $D = 1.36$ . This optimal walking gait has

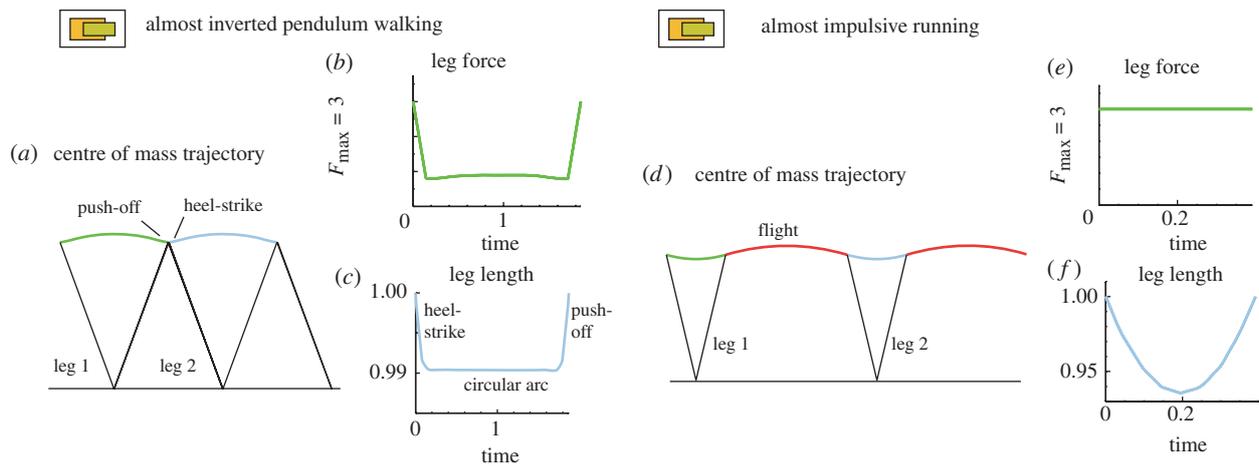


Figure 4. (*a–c*) A non-impulsive analogue of an inverted pendulum walking with the springless telescoping model, minimizing the work cost  $C_w$  at  $V = 0.39$ ,  $D = 1.36$  and  $F_{\max} = 3$ . The gait consists of the body moving in circular arcs vaulting over a straight leg, with the transition from one circular arc to the next accomplished by finite-force impulses (with  $F \sim F_{\max}$ ) by the trailing leg and the leading leg, identified with the push-off and the heel-strike, respectively. (*d–f*) Optimal running with springless telescoping model, minimizing the work cost  $C_w$  at  $V = 1.2$ ,  $D = 2.8$  and  $F_{\max} = 3$ . The running stance phase has the leg force at  $F_{\max}$  identically. The graphs corresponding to the two legs' stance phases are overlaid in (*b, c, e, f*) and they appear indistinguishable, indicating left–right symmetry.

negligible double-support phases, despite allowing for them. A substantial double support is optimal only when the maximum leg force available is too close to or less than 1, when a single leg is unable to support body weight. For most of the stance phase, the leg length is constant and no leg work is performed. Negative work is performed at the beginning of the stance phase, akin to a heel-strike. Positive work is performed at the end of each stance phase, akin to a push-off.

Figure 4*d–f* shows the non-impulsive analogue of impulsive running, found to be optimal at  $V = 1.2$  and  $D = 2.8$ . Here, the leg force is equal to the maximum allowed leg force throughout the stance phase. Negative work is performed for the first half of the stance phase, and positive work during the second half, redirecting the centre of mass velocity from downward to upward.

The energy cost landscape appears to be relatively flat near the transition from walking to running. Figure 5*a–c* shows the non-impulsive analogue of pendular running [5], optimal at  $V = 0.9$  and  $D = 1.17$ . Figure 5*d–f* shows a very different asymmetric gait, but with a cost only 1 per cent greater than that of the optimal pendular run at the same speed and stride length. The energy cost landscape in the  $[V, D]$  space with the finite leg forces is probably similar qualitatively to when the optimal gaits involved idealized impulses in Srinivasan & Ruina [5] and Srinivasan [30], so it is not plotted here.

**Adding a force–velocity relation.** First, recall that all three optimal gaits obtained above with only a simple bound on the leg forces made use of the maximum available forces available to them at some point during the stance phase. So by continuation, when a non-trivial force–velocity relationship is imposed for the telescoping actuator, one still expects the optimal gaits to take advantage of the maximum available forces. However, now, the maximum available forces are dependent on the leg length rate—in particular,

the maximum available force when the leg is performing negative work is greater than when the leg is performing positive work. This suggests that the leg force will cease to be symmetric about mid-stance. Figure 6*a–c* shows the effect of a linear force–velocity relation on the walking gait of figure 4*a–c*. As just argued, the leg force is slightly asymmetric about mid-stance—the push-off is slightly different from the heel-strike—but this difference has little effect on the centre of mass trajectory, which remains quite close to circular arcs. Similarly, figure 6*d–f* shows the optimal running gait ( $V = 1.2$ ,  $D = 2.8$ ) when constrained by a linear force–velocity relation. The leg force hugs the force–velocity constraint, just as when there was a simple bound on the force, the leg force was identically equal to the maximum force.

#### Minimize generalized work costs $C_g$ and $C_{AM}$ .

As described in §5, the quasi-steady model  $C_{AM}$  with a convex  $g(v)$  is a special case of  $C_g$ . Minimizing  $C_{AM}$  with a force–velocity relationship ( $V = 0.39$ ,  $D = 1.38$ ) gave the gait depicted in figure 7*a, b*, again very similar to inverted pendulum walking.

In addition to  $C_{AM}$ , I considered two more ad hoc examples of  $C_g$ . Figure 7*c, d* shows the optimal walking gait at  $V = 0.39$ ,  $D = 1.38$  for  $C_g$  with an  $g(v) = v^2$  and figure 7*e, f* shows the optimal walking gait for  $g(v) = (v - 0.5)^2$ . Both are close to inverted pendulum walking. Numerical experiments with a few other convex and positive  $g(v)$  also result in qualitatively similar optimal gaits to that which minimizes the work cost  $C_w$ , in particular, inverted pendulum walking at low speeds. At higher speeds, running gaits (not shown) that involve leg forces that essentially reach the force bounds, be they simple force bounds or force–velocity relations, appear to be optimal. In this sense, these gaits are analogous to impulsive running. As before, when the leg force tracks the force–velocity relation, the leg force can be asymmetric about mid-stance. Overall, these results suggest that

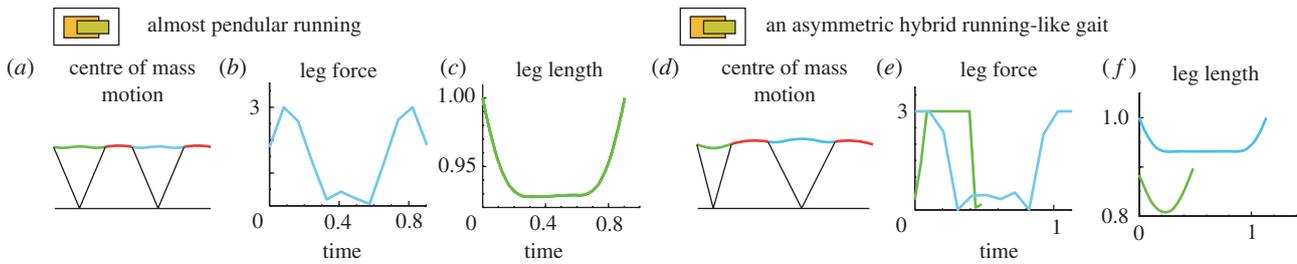


Figure 5. Two near-optimal gaits at  $V = 0.9$ ,  $D = 2.34$  and  $F_{\max} = 3$  for the springless telescoping biped. (a–c) The non-impulsive analogue of pendular running. (d–f) A hybrid asymmetric gait, with one stance phase pendular and the other stance more compliant. The cost of the second gait is only about 1 per cent more than that of the first gait.

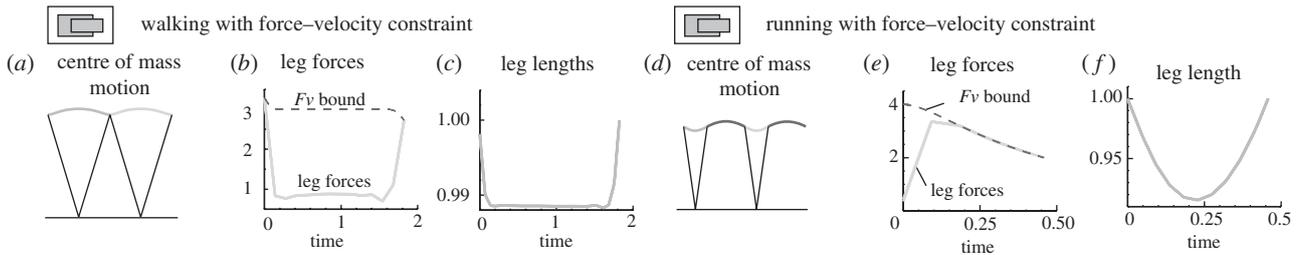


Figure 6. Optimal gaits when the springless telescoping actuator has a linear force–velocity relation  $F \leq F_0(1 - v/v_{\max})$  with  $F_0 = 3$ , and  $v_{\max} = 2$ . (a–c) Walking gait minimizing the work cost  $C_w$  at  $V = 0.39$  and  $D = 1.34$ , close to inverted pendulum walking. (d–f) Running gait minimizing the work cost  $C_w$  at  $V = 1.2$  and  $D = 2.8$ . The leg force mostly takes advantage of the maximum allowed forces by the force–velocity relation, shown as a dotted line in (b,e). The plots corresponding to the two legs are overlaid and are indistinguishable in these plots, suggesting symmetry.

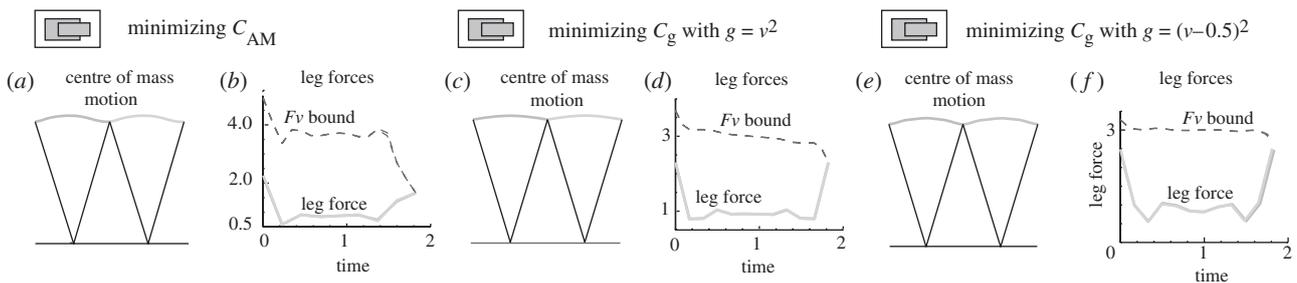


Figure 7. (a,b) Walking with the springless telescoping model, minimizing the convex empirical cost  $C_{AM}$  at  $V = 0.39$ ,  $D = 1.36$ ,  $F_0 = 3$  and  $v_{\max} = 0.8$ . (c,d) Walking with the springless telescoping model at  $V = 0.39$  and  $D = 1.38$ , minimizing  $C_g$  with convex  $g(v) = v^2$ . (e,f) Minimizing  $C_g$  with  $g(v) = (v - 0.5)^2$ . All these three optimal walking gaits are quite close to inverted pendulum walking and each other.

the optimal gaits for  $C_g$  are essentially similar to that for  $C_w$ .

**With additive force costs.** Consider a linear force penalty (i.e.  $C_{F\beta}$  with  $\beta = 1$ ) added to the costs  $C_w$ ,  $C_{AM}$  or  $C_g$ . Then, the new metabolic power would be  $Fg(v) + b_1 F = F(b_1 + g(v))$ , which is also of the form  $C_g$  with a  $g(v)$  simply shifted up by the constant  $b_1$ , still positive and convex. So, it is not surprising that essentially no qualitative changes are found in the optimal gaits for a linear additive force cost.

A quadratic force cost  $C_{F2}$  is qualitatively different, being a much stronger penalty on large forces and, less directly, force fluctuations. See appendix A for a simple example that informs the following discussion. Periodic locomotion, as a task, does not prescribe the average leg force—but periodicity does specify that the average vertical force on the body must equal the weight. At small stride lengths, when the legs do not make large angles with the vertical, this vertical force

constraint is similar to an average leg force constraint. Thus, when one has a cost like  $C_{F2}$ , one obtains gaits in which both the force fluctuations and the actual leg forces are minimized. The leg forces are reduced by having large double-support phases and the force fluctuations are reduced by having the leg forces relatively constant. In fact, at small step lengths, the optimal walking gait (not shown) tended to have mostly double support and very little single stance, an artefact that can perhaps be mitigated by having a substantial leg-swing cost. But even for a pure  $C_{F2}$  cost, it appears that for high enough speeds and step lengths, the optimal gait, at least locally, is one with a flight phase (figure 8).

**With explicit bounds on force rates.** Thus far, it has been assumed that the leg force can rise from zero instantaneously if necessary. But muscles have internal dynamics that limit the rate of increase of the force they produce. To simply model this, I bound the rate of leg

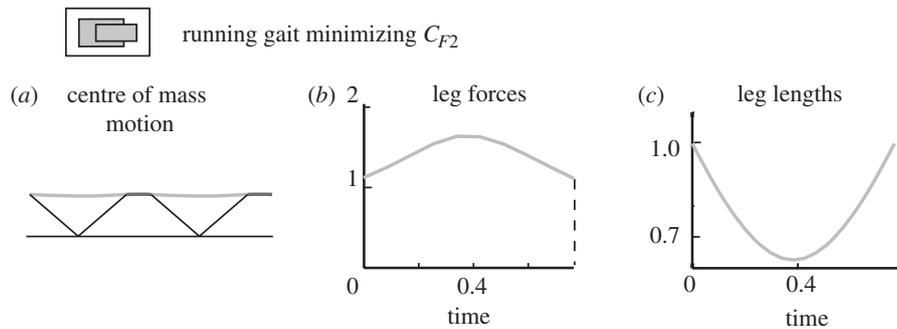


Figure 8. Running gait that minimizes the  $C_{F2}$  cost for the springless telescoping biped at  $V = 2$ ,  $D = 3.78$ .

force change:  $-3 \leq \dot{F} \leq 3$ , the specific values selected so the leg force rate bound is limiting, but not so limiting that a gait cannot be supported. It is further assumed that the forces must start and end at zero, and can change only continuously. Figure 9*a–d* shows the results of minimizing  $C_w$  at  $V = 0.39$  and  $D = 1.36$  with this force rate constraint. Here, the presence of double support is energetically optimal—that is, making the double-support phase zero increases the energy cost. Note that this double support is an explicit effect of bounding the force rates. Exactly the same minimization, in the absence of bounding the force rates, showed that having *no* double support was optimal (figure 4). Figure 9*e* shows the even smoother (and realistic) optimal leg forces when the second derivatives of the leg forces were appropriately bounded.

At higher speeds, a running gait with a unimodal (increasing–decreasing) leg force profile is obtained, with the increase and decrease in the leg force happening at the maximum available leg force rate (figure 10*a–c*). If the second derivative of the leg force is bounded instead, a running gait with a smoother (piecewise quadratic) unimodal leg force  $F(t)$  is obtained (figure 10*d–f*). Interestingly, for a range of such rate bounds, a graph between the leg force and the leg compression looks approximately linear—the so-called linear spring-like leg behaviour [9,10]—despite the presence of no springs in the calculations [53].

## 8. SPRINGLESS KNEED BIPED

The springless kneed biped is shown in figure 1*b*. The knee angle is  $\alpha$  and the knee moment is  $M$ , positive when anti-clockwise on the upper leg. The leg length is then  $l = \cos(\alpha/2)$  and the leg length rate  $\dot{l} = -\sin(\alpha/2) \cdot \dot{\alpha}/2$ . The compressive leg force  $F$  and the knee moment  $M$  are related as:  $M = -F \sin(\alpha/2)$ . Thus, when  $\alpha$  is small, a large leg force ( $F = -M/\sin(\alpha/2)$ ) is possible with a small leg moment; also, although not explicitly accounted for by this formula, finite leg forces are possible at  $\alpha = 0$  with zero leg moments. A related singularity is that the knee angular velocity  $\dot{\alpha}$  goes to infinity for non-zero leg length rate  $\dot{l}$  when  $\alpha \rightarrow 0$ ; see the electronic supplementary material for how these ‘kinematic singularities’ are regularized

during the optimization. The leg power is, of course, identical to the knee power:  $P = F\dot{l} = M\dot{\alpha}$ .

**Minimizing work cost  $C_w$ .** As just mentioned, the mechanical power  $P = F\dot{l}$  of the leg is identical to the mechanical power  $M\dot{\alpha}$  of the knee muscle in the springless kneed biped. Therefore, the results of minimizing the work cost  $C_w$  for the kneed biped of figure 1*b* are identical to those obtained when  $C_w$  is minimized for the telescoping biped of figure 1*a*, in the absence of other constraints. Force constraints are now most naturally imposed on the knee moment (and, therefore, the muscle force) rather than the leg force. A simple bound on the knee moment, as long the bound is not too small, results in essentially the classic work-minimizing gaits from earlier (inverted pendulum walking, etc).

**Minimizing ‘robot’ knee moment cost  $C_{F2}$ .** As noted, the kinematic singularity at the knee allows a kneed biped to exert a finite leg force with very little knee moment when the leg is close to straight. Thus, when the metabolic cost is dominated by the  $C_{F2}$  term (here  $\int M^2 dt$ ), one might superficially expect that the biped would operate near the kinematic singularity for as much of the stance phase as possible. Indeed, this is true of the optimal walking gait, for instance, as shown in figure 11 for  $V = 0.39$  and  $D = 1.38$ . The leg is fully extended for almost all the stance, except for the beginning and the end of the stance phase, when push-offs and heel-strikes happen with large leg forces. In the limit of infinite allowed leg forces, this walking gait appears to approach the classic inverted pendulum gait described earlier. At higher speeds, it appears advantageous to break into a run (figure 12) and as the allowed leg forces are increased, the optimal gait approaches the impulsive running gait, even though the leg force seems somewhat U-shaped (figure 12*b*). Thus, remarkably, this force-based cost  $C_{F2}$ , which is completely unrelated to the work cost  $C_w$ , gives essentially the same optimal gaits for this kneed biped, at least in the impulsive limit.

**With a force–velocity relation.** The force–velocity relation is now a constraint on the knee moment  $M$  as a function of the knee angular velocity  $\dot{\alpha}$ . The simplest such constraint is linear. Noting that a positive  $\dot{\alpha}$  corresponds to the knee muscle stretching, the linear force–velocity relation used is

$$M \leq M_0 \left( \frac{1 + \dot{\alpha}}{\omega_{\max}} \right) \quad \text{when } \dot{\alpha} \geq -\omega_{\max}, \quad (8.1)$$

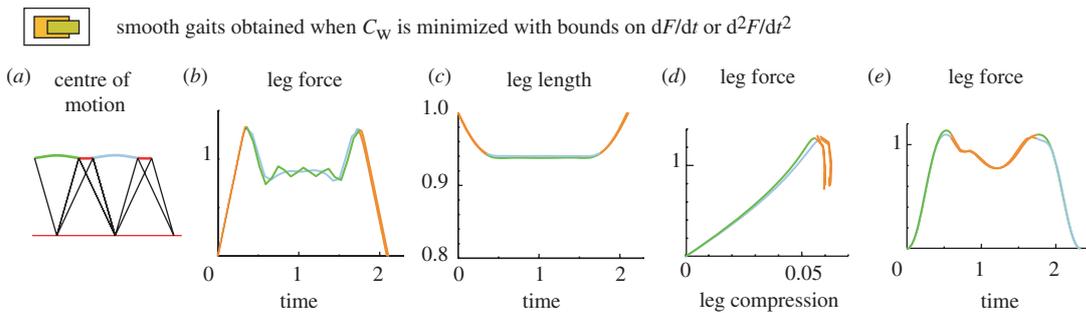


Figure 9. (a–d) Walking gait that minimizes the work cost  $C_w$  for the springless telescoping biped, in the presence of bounds on the leg force rates, at  $V = 0.39$  and  $D = 1.38$ ,  $-4 \leq \dot{F} \leq 4$ . (e) Leg forces for the optimal gait when the second derivative of leg force is bounded:  $-20 \leq \ddot{F} \leq 20$ , without constraining  $F$  or  $\dot{F}$ . It is found that a non-zero double support becomes optimal and using the maximum available leg force is no longer optimal. The specific bounds on  $\dot{F}$  and  $\ddot{F}$  determine the amount of double stance and the maximum leg forces that are optimal. The slight wiggles in (e) are an indication of near but imperfect convergence.

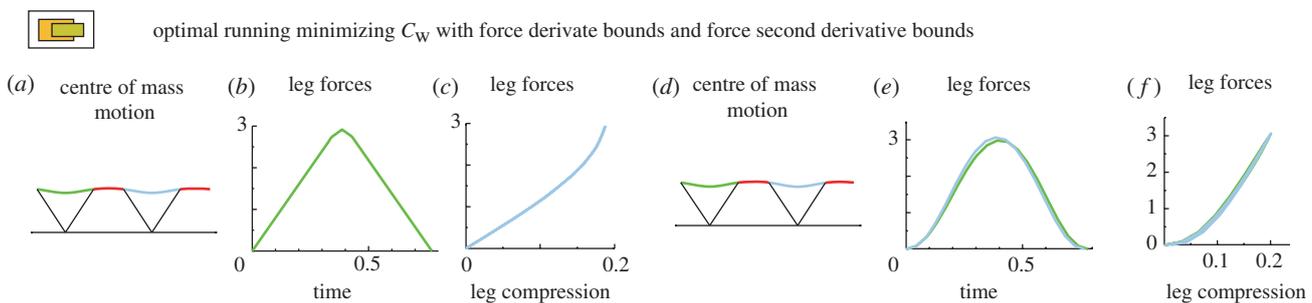


Figure 10. Running gait that minimizes the work cost for the springless telescoping biped at  $V = 0.39$  and  $D = 1.38$ , (a–c) in the presence of bounds on the leg force rates ( $-3 \leq \dot{F} \leq 3$ ) and (d–f) in the presence of bounds on the second derivative of forces ( $-100 \leq \ddot{F} \leq 100$ ). Both optimizations assumed a leg force bound as well ( $F \leq 3$ ). It is found that the leg force as a function of the leg compression, while nonlinear, is not far from linear on average, suggesting that the so-called linear spring-like leg behaviour at optimality requires no springs.

and  $M \equiv 0$  when  $\dot{\alpha} < \omega_{\max}$ . Again,  $M_0$  and  $\omega_{\max}$  were selected such that they support gaits, but are not too high, as can be gleaned from the figures referred to below.

Because of the kinematic singularity, at the end of stance, if  $l \approx 1$  and  $\dot{l} > 0$ ,  $\dot{\alpha}$  is negative and very large, the available knee moment  $M$  is zero, because of the force–velocity constraint equation (8.1), and unable to provide the necessary compressive leg force for the push-off. Thus, introduction of a force–velocity constraint implies that the biped might have to operate slightly away from the kinematic singularity, so that reasonable knee moments are available. Figure 13a–f shows the  $C_{F2}$ -minimizing walking gait, when the biped is explicitly constrained to operate away from the kinematic singularity by setting  $l_{\max} = 0.95$  (and an extra leg force constraint  $F_{\max} = 3$ ). The knee moment required now is well within the feasible set allowed by the force–velocity constraint. Figure 13g–l shows a  $C_w$ -minimizing running gait in the presence of force–velocity constraints on the knee muscle, constrained by  $F_{\max} = 3$ . When the force–velocity constraint is active, a slight asymmetry probably appears about midstance as for the telescoping model.

**Minimizing  $C_g$  costs.** In limited numerical explorations, minimizing  $C_g$ , including  $C_{AM}$  and two other instances of  $C_g$  with  $g(v) = \dot{\alpha}^2$  and  $(\dot{\alpha} - 0.5)^2$ , gave rise

to inverted pendulum walking and running-like gaits at low and high speeds, respectively.

## 9. SPRINGY TELESCOPING BIPED

Animals have compliant tendons in series with muscles. Such tendons can substantially affect the energetics of steady locomotion by storing work performed on it and returning it later during the stance. Figure 1c is perhaps the simplest biped model that has both a muscle (telescoping actuator) and a tendon (linear springs). The springs are linear, perfectly elastic and have no damping.

As a practical matter, because the muscle length rate and therefore the metabolic rate function depends directly on the muscle force rate for these springy bipeds (unlike the springless bipeds), the muscle force rates are bounded during the optimizations. In general, however, the force rate bounds were chosen to be so large ( $-20 \leq \dot{F} \leq 20$ ) that they are never taken advantage of in most of the optimal solutions described below. In this sense, it is as if no force rate bounds are used for most of the optimizations below, unless noted otherwise. Note on the other hand that most of the optimizations with the springless bipeds (except for one) involved no explicit force rate bounds.

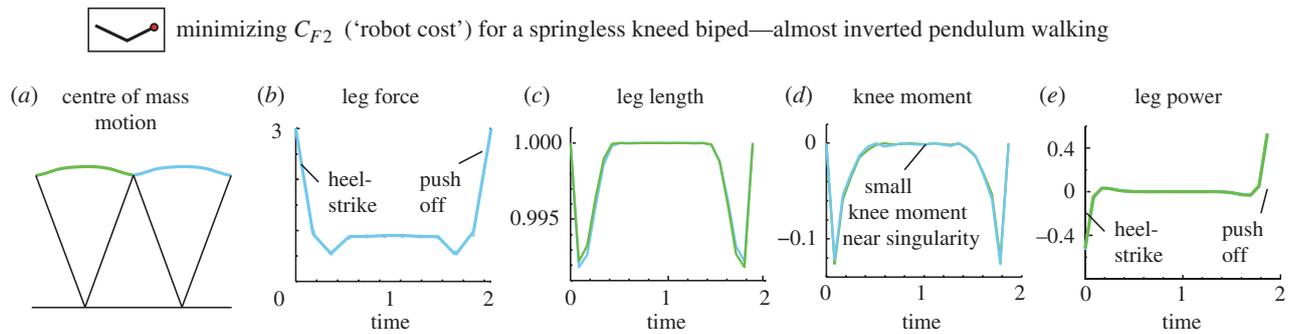


Figure 11. Walking gait that minimizes the  $C_{F2}$  cost for a springless knee biped with no force–velocity constraint, at  $V = 0.39$ ,  $D = 1.38$  and with  $F_{\max} = 3$ . Again, this gait is quite close to inverted pendulum walking, with no double support.

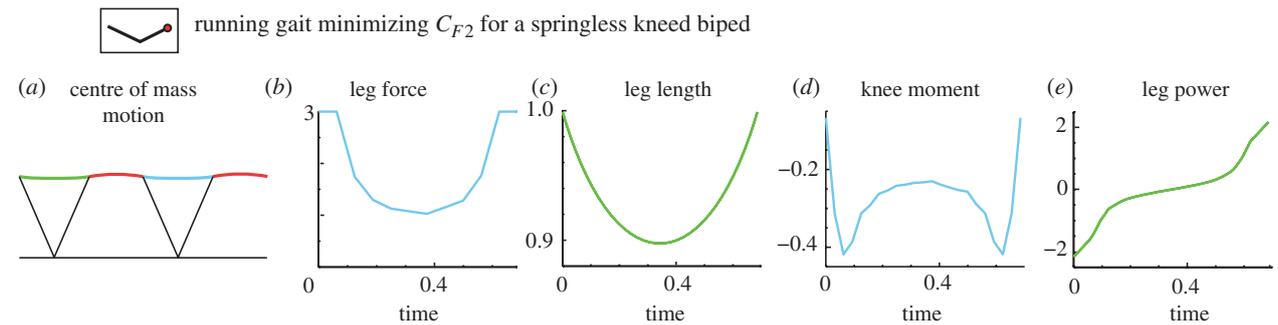


Figure 12. Running gait that minimizes the  $C_{F2}$  cost for a springless knee biped with no force–velocity constraint, at  $V = 1.2$ ,  $D = 2.8$  and with  $F_{\max} = 3$ . When the maximum allowed forces is increased, the running gait at this  $[V, D]$  approaches impulsive running, although the leg force pattern is still U-shaped.

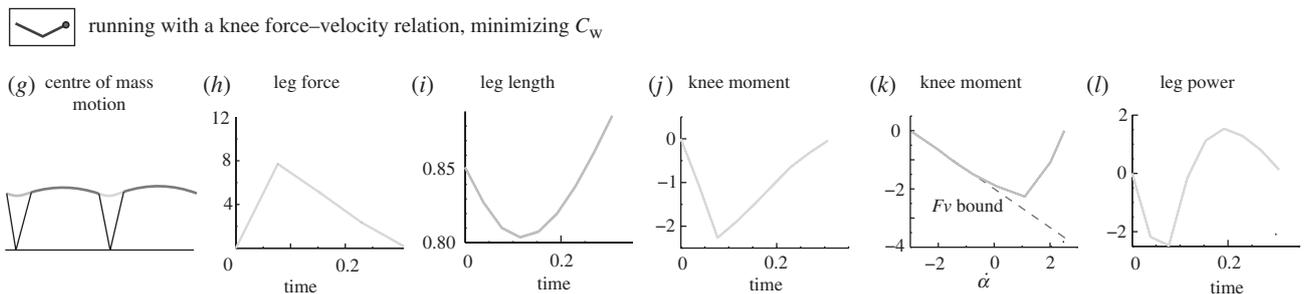
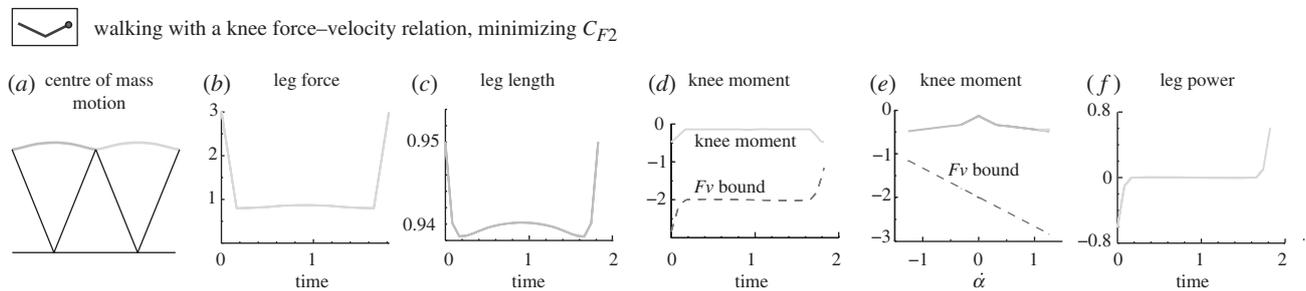


Figure 13. (a–f) Walking gait that minimizes the  $C_{F2}$  cost for springless knee biped with a linear force–velocity constraint on the knee moment, when  $V = 0.39$ ,  $D = 1.38$ ,  $l_{\max} = 0.95$  and  $F_{\max} = 3$ . Again, this gait is close to inverted pendulum walking. (g–l) Running gait that minimizes  $C_w$  for the springless knee biped at  $V = 1.2$  and  $D = 2.8$ , with a force–velocity relationship for the knee muscle. This gait shows more time-reversal asymmetry in the trajectory and the leg force profile than other gaits here on account of the force–velocity relation.

**Minimizing work cost  $C_w$ .** Given the same constraints including leg force capabilities, this springy biped can perform *every* gait that can be performed by the springless biped of figure 1a and vice versa. Of particular interest are gaits that minimize the muscle

work cost  $C_w$  for the springy biped. Figure 14 shows one such gait, with essentially zero work done by the actuator. All the positive and negative work is done by the series spring, while the series telescoping actuator remains at constant length (isometric contraction)

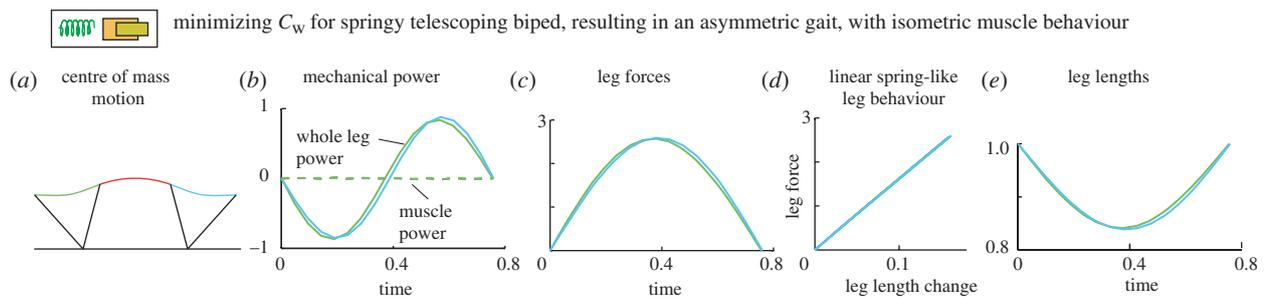


Figure 14. Springy telescoping biped: a gait that minimizes  $C_w$  for the springy telescoping model with  $V = 1$  and  $D = 2.4$ . The muscle work is essentially equal to zero. This gait may be somewhat reminiscent of skipping gaits [11].

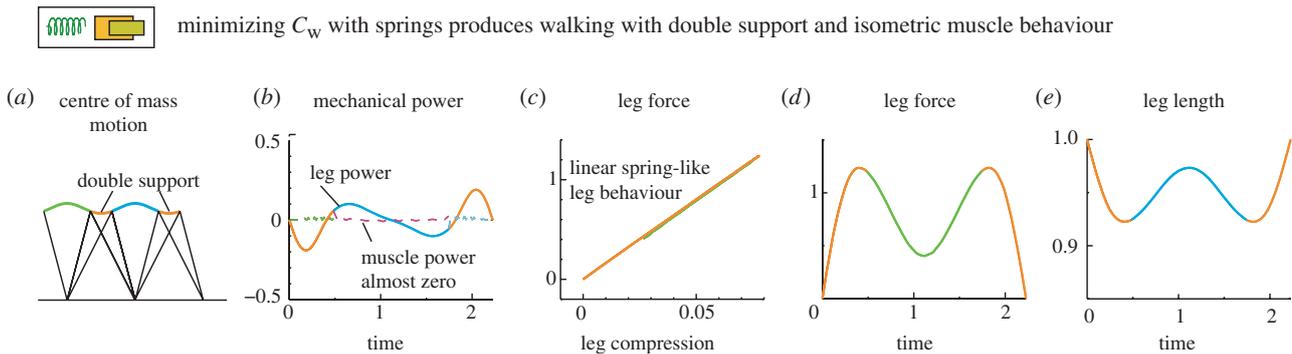


Figure 15. Walking with a springy telescoping model, with a relatively soft spring: a non-dimensional stiffness of 16, at  $V = 0.39$  and  $D = 1.38$ , minimizing  $C_w$  with enforced left–right symmetry. The leg force rate is bounded:  $-20 \leq \dot{F} \leq 20$ , but the bound is never used. Essentially, zero mechanical work is performed by the muscles and all the mechanical work is performed by the springs in series. The leg force versus leg compression behaves, not surprisingly, in a linear manner.

and provides a force that is at all times equal to the spring's. The stiffness of the spring was chosen to be close to the 'apparent stiffness' of a human leg while running (non-dimensional stiffness is 16), but the actual value does not affect the qualitative conclusions, as long as the muscle is capable of sufficiently high forces and the stiffness is not too small ([12]; apparent stiffness is defined as the slope of the leg force versus leg compression curve and is not a true stiffness). The bound on the leg force was 3, higher than that required by the work-free gait—that is, this force bound is not used by the optimal gait, just as the force rate bound is also unused by the optimal gait.

The work-free gaits ( $C_w = 0$ ) of this biped are identical to the periodic gaits of the so-called spring-mass biped model [10,12,54,55], which has only a telescoping spring and no series actuator. For such unactuated spring-mass models, it can be shown that for a given speed, step length and spring stiffness within some bounds, there usually exists (that is, it is non-exceptional to find) one or more isolated walking and/or running gaits with left–right symmetry; see [12] and [30] for counting arguments. For an example of one such symmetric and muscle-work-free running gait, see figure 16 (although it was not obtained by minimizing  $C_w$  as explained below). If one allows that the left and the right stance phases need not be identical, there is a one-parameter family of work-free gaits—and therefore infinitely many equally optimal gaits—for specified speed, stride length and spring stiffness. The gait shown in figure 14 is one such asymmetric

gait. Note that the stance phases for this gait are time-reversed versions of each other. The optimization converged to an asymmetric gait perhaps because the symmetric gait is a special case and asymmetry is generic (on the other hand, asymmetry seems 'structurally unstable' as noted below).

Figure 15 shows an essentially muscle-work-free walking gait for the springy telescoping biped; left–right symmetry was imposed for this optimization for simplicity and to emphasize similarity with human walking. One obtains a qualitatively similar walking gait—with double-humped leg force profile—for a range of spring stiffnesses.

**Minimize  $C_w$  with a force penalty.** That work-free gaits are obtained when  $C_w$  is minimized is to be expected, and is not surprising. But what if there is a strong cost penalty for muscle forces? Figure 16 shows a running gait that minimizes a linear combination of work cost  $C_w$  and a force cost  $C_{F2}$ :  $\lambda C_w + (1 - \lambda) C_{F2}$ , for  $\lambda = 0.9$ . The muscle work performed is still essentially equal to zero, despite the metabolic cost having a substantial force cost. Thus, the same muscle-work-free gait (figure 16) minimizes the cost for both  $\lambda = 1$  and  $\lambda = 0.9$ . More generally, it appears that the work-free gait is optimal for a range of  $\lambda$  near 1, another instance of the insensitivity of the optimum to changes in the cost function. This particular insensitivity of the optimum is explained in appendix D as being related to the non-smoothness of the cost at the optimum at  $\lambda = 0$ . Also, note that the gait in figure 16 corresponding to  $\lambda = 0.9$  is symmetric, while the optimal gait (figure 14)

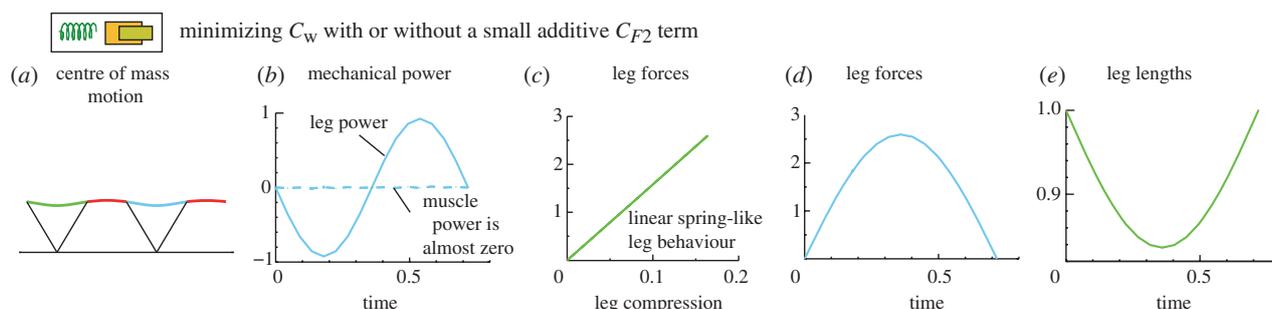


Figure 16. Springy telescoping biped: running gait minimizing  $\lambda C_w + (1 - \lambda) C_{F2}$  for  $\lambda = 0.9$ ,  $V = 1$  and  $D = 2.4$ . Despite a substantial cost for isometric force, the optimal gait appears to still minimize  $C_w$ , requiring essentially zero work. The leg force versus leg compression shows a linear spring behaviour, as would be expected.

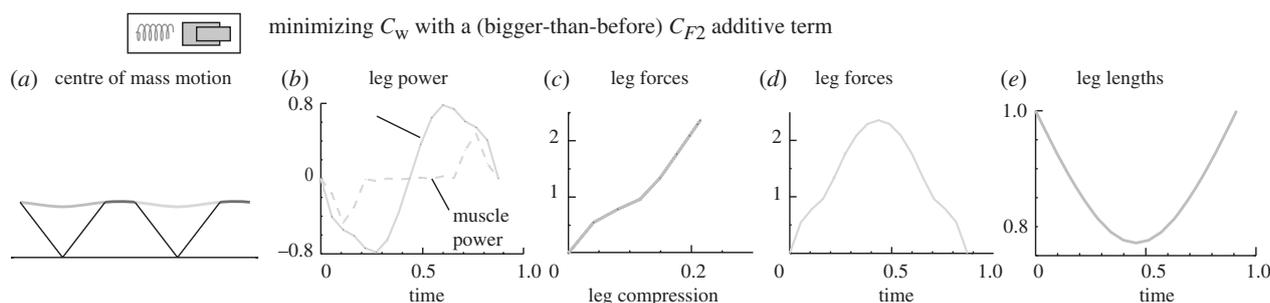


Figure 17. Running with the springy telescoping biped. Minimizing  $\lambda C_w + (1 - \lambda) C_{F2}$  for  $\lambda = 0.75$ ,  $V = 1$  and  $D = 2.4$ . As the force penalty is increased, the optimal gait moves away from being completely work-free. Negative and positive work is performed by the muscle.

corresponding to  $\lambda = 1$  was generically asymmetric. It appears that the presence of the small force penalty  $C_{F2}$  makes symmetry optimal, while  $C_w$  alone has both symmetric and asymmetric optimal gaits. Figure 17 shows the optimal running gait for  $\lambda = 0.75$ , so that the weight on  $C_{F2}$  is higher than when  $\lambda = 0.9$ . This gait does involve some muscle mechanical work, even though work-free gaits exist, as demonstrated by figures 14 and 16. Thus, it seems like the work-free gaits are optimal for a range of  $\lambda$  near 1, but not optimal for lower  $\lambda$ , when the force penalty starts dominating the nature of the optimum.

**Minimize  $C_g$  costs.** As §6 would suggest, minimizing  $C_g$  also results in a zero  $C_w$  gait. This is true whether or not even when there is a cost for isometric force (force with no work). As before, I used a few different  $g(v)$ , including  $v^2$ ,  $(v - 0.5)^2$  and the  $g(v)$  for  $C_{AM}$ . When  $C_g$  has a cost for isometric force, there is a tendency toward left–right gait symmetry.

**High stiffness limit.** If the leg force is not explicitly constrained, work-free gaits exist for any high stiffness of the linear tendon spring. At very high stiffnesses, the work-free running gait looks like the impulsive running gait, with parabolic free flights interrupted by brief bounces on the leg springs, with the series telescoping actuator remaining isometric.

However, such work-free gaits involve high leg forces and become infeasible when the leg forces or the leg force rates are bounded. Indeed, these force and force rate bounds become active constraints mainly in the high tendon stiffness limit. When the force rates are bounded and the optimal gaits are sought for given speed and stride length, the high stiffness limit

approaches the limit of having no springs, namely the springless telescoping biped. Figure 18 shows the optimal walking gait at  $V = 0.39$  and  $D = 1.38$  when the stiffness is high ( $k = 500$ ). Note that the gait is close to inverted pendulum walking, which is optimal in the absence of the spring, except made a little smoother by the presence of the spring. At a higher speed ( $V = 1.2$  and  $D = 2.4$ ), one obtains a running gait, in which the maximum available force rates were used, analogous to impulsive running.

## 10. SPRINGY KNEED BIPED

Similar to the springy telescoping biped, the springy kneed biped has work-free gaits in which the muscle remains exactly isometric throughout the stance phase, while the springy tendon performs negative work first (stretches) and positive work next (shortens), redirecting the centre of mass velocity from down to up. When there are no constraints on the forces and  $C_w$  is minimized, the optimization converges to one of the work-free gaits, corresponding to the specified speed and stride length. As before, work-free gaits continue to be optimal when the cost is slightly perturbed as  $\lambda C_w + (1 - \lambda) C_{F2}$  for a range of  $\lambda$  near 1. Also, the work-free gaits were found to be optimal for  $C_{AM}$  and for a couple of other  $C_g$ 's that were minimized. For example, figure 19 shows the gait-minimizing  $C_{AM}$  (with the constant 0.5 added to its  $g(v)$ ) and is essentially muscle-work-free. It is interesting to note that despite the kinematic singularity, the relation between leg force and leg length change is not far from linear.



torque essentially decreasing linearly from an accelerating torque to a decelerating torque, in a manner that satisfies the swing time and swing amplitude constraint.

These additive costs were used with appropriate pre-factors in a couple of springless optimizations described so far, but the optimal gait did not change qualitatively when the speed and stride lengths were fixed. The leg-swing cost does affect the duty factor by imposing a penalty on too short a swing time, or conversely, too great a double-support phase. But inverted pendulum walking and impulsive running already have as great a swing time as possible for the given type of gait (walking versus running), with all swing times being equal by symmetry. Perhaps, this is the reason for the robustness of such optima to such additive swing costs.

To be sure, the separation of the total cost into additively separable stance and swing costs is a simplification. For instance, work performed to move the hip around (here called the stance cost) can potentially contribute to leg swing, making the separation into stance and swing costs, as defined above, generally impossible. But considering the leg swing as being separable for the purposes of understanding the structure of the optimal solution is supported in some cases by optimization results from a model with massive legs. I considered a model similar to figure 1*a*, except for legs having hip muscles and point-masses at their feet. I found that the  $C_w$ -minimizing gait was inverted pendulum walking with push-offs, heel-strikes and vaulting on circular arcs. And the leg swing was accomplished by hip impulses at the beginning and end of the swing. Thus, the work optimal gait for a model with massive legs consists of *pastng together* the optimal gait for a model with massless legs and the optimal leg-swing strategy with the legs considered in isolation. This separability of the structure of the optimal gait is probably untrue for strange animals whose body mass is dominated by their leg mass, especially if most of the leg mass is close to the feet.

## 12. DISCUSSION: GENERAL OBSERVATIONS ABOUT THE OPTIMAL LOCOMOTION

The following observations about the mechanics and energetics of legged locomotion arise from the many calculations detailed in this paper and other general considerations. Many of these observations correspond to salient features of human terrestrial locomotion that the hypothesis of energy optimality is able to explain.

**1. Unavoidability of a minimum amount of leg or joint work.** In periodic legged locomotion on level ground, the total mechanical energy of the animal is the same at the beginning and the end of a gait period. Despite such overall energy neutrality, common experience and careful experiments tell us that our legs perform substantial work when we walk and run. Is such leg work unavoidable for legged locomotion? If so, why?

One reason for performing work is dissipation. However, the springless telescoping biped considered here

had no dissipative mechanisms—no collisions and had only no-slip frictional contact.<sup>2</sup> By minimizing  $C_w$  for the springless telescoping biped, I have shown—here and in Srinivasan & Ruina [5]—that travelling at non-infinitesimal speeds and step lengths requires non-infinitesimal positive leg work (that is, the optimal  $C_w$  does not go to zero) even in the absence of frictional or collisional dissipation, as long as the biped's body is constrained to never penetrate the ground. This minimum required positive leg work, given the lack of physical dissipation, must be exactly cancelled by the performance of an equal amount of negative leg work elsewhere during the gait.

Now imagine a more realistic biped—say, with more body segments, especially legs, with masses—with frictionless joints, slip-free frictional contact with the ground and plastic mechanical collisions that can in principle be avoided by stepping carefully. For such a more realistic biped, I believe (based on unpublished calculations and some arguments with additive leg-swing costs) that this work-unavoidability claim is even stronger: travelling at non-infinitesimal speeds requires non-infinitesimal leg work, or more generally, some joint work. That is, the condition of non-infinitesimal step lengths from the previous paragraph is unnecessary. See [56] for a related result for the special case of unactuated downhill walking 'passive dynamic' robots.

A potential source of confusion might be that in real human or robot walking, heel-strike collisions do appear to be a big source of energy loss, which must be made up by positive work elsewhere [4,57–59]. But this heel-strike collisional loss may be interpreted as just the animal's way of performing negative work (which is unavoidable), without using muscles and at the optimal time, i.e. at the end of stance, as in our optimal inverted pendulum walking gaits. In other words, if the motion involves collisions, the biped has at least that much energy to replace by positive work; and if the biped avoids collisions, the legs have to do both positive and negative work.

Two more clarifications are in order. First, the above claims are with respect to 'leg or joint work', as opposed to 'actuator or muscle work'. Of course, locomotion is possible in principle with just spring work—as in Geyer *et al.* [10], O'Connor & Kuo [11] and Srinivasan & Holmes [12], and the springy bipeds here—without the series muscles performing work, but the spring work is part of the unavoidable leg work. Second, it has been conjectured by ([60]; also personal communication 2008) that it is possible to construct bipeds that have legs with closed kinematic chains that can overcome the unavoidability of leg work. My claim about the unavoidability of leg work does not apply to these unconventional bipeds; another such unconventional biped is McGeer's 'silly wheel' [61].

Roughly, the origins of this leg/joint work unavoidability is related to the assumption of (non-infinitesimal) finite gravity. In zero gravity, locomotion

<sup>2</sup>If the surface was frictionless, the biped could glide without performing work once motion is initiated somehow, say by pushing off a wall.

once begun—perhaps by pushing off a wall—can continue for ever with no additional leg work and therefore no cost, coasting ad infinitum. In finite gravity, at least for the simple biped models here with non-infinitesimal step lengths, the only two ways of going forward performing no work for extended periods of time during a single step is to either lock the leg joints (as in an inverted pendulum phase) or have zero leg force (as in a flight phase). Both these phases have a body trajectory that is strictly concave-downwards ( $d^2y/dx^2 < 0$ ) because of the gravitational pull. But a periodic trajectory has to be horizontal on average, so it appears that the leg needs to do some finite amount of work to make the average value of  $d^2y/dx^2$  equal zero. It remains an open mathematical problem to prove this intuition rigorously for any reasonably realistic class of bipeds with leg mass.

**2. Optimality of inverted pendulum walking and impulsive running.** Inverted pendulum walking and impulsive running are classic idealized descriptions of human walking and running, going back at least half a century [6,62]. Srinivasan & Ruina [5] showed that these originally subjective idealizations have a more objective existence as the work-minimizing gaits of the same simple biped models (figure 1*a,b*). These classic descriptions appear to minimize a vast number of cost functions for the models in figure 1*a,b* and the stiff-tendon limit of the models in figure 1*c,d*. For instance, this optimality result has been extended to minimizing a more general cost  $C_g$ , including the empirical cost  $C_{AM}$ . These gaits also appear optimal for a vastly different cost function, namely the integral of squared knee moment  $M^2$ , if the biped has a knee.

**3. Double-humped vertical ground reaction force for walking.** The vertical ground reaction force (GRF) for human walking has a double-humped (bimodal) profile. Somewhat analogous to this bimodal GRF, idealized and impulsive inverted pendulum walking has two large force spikes at the beginning and the end of a stance phase, and much smaller forces during the pendular part of the stance phase. When bounds are placed on how fast the leg force can increase from zero, as exists for a realistic muscle, the force spikes at the ends of inverted pendulum walking become smoother (figure 9), making the optimality of a bimodal GRF more apparent. Somewhat interestingly, the  $C_g$ -optimal gaits for the springy telescoping model also had a double-humped vertical GRF, for a range of tendon spring stiffnesses, from  $k = 16$  to 500, at least for the specific  $V = 0.39$  considered in detail. But note that for the springy models, work-free gaits with no flight phase but with a unimodal vertical GRF are also possible for some leg spring stiffnesses, similar to the ‘grounded running’ gaits of cockroaches and some birds [11,12].

**4. ‘Bang-coast-bang’ structure of the  $C_g$ -minimizing gaits in springless models.** The optimal gaits for  $C_g$  for the springless models all had a simple structure. In these gaits, the mechanical power was close to zero for most of the time, either in pendular phases with  $\dot{l} = 0$ , or in free flight with  $F = 0$ . Thus, the gait is mostly energetically passive, with mechanical work being performed in brief episodes of large leg forces. Such a strategy is sometimes called

bang-coast-bang by optimal control theorists [63]. In [5], where this structure was discovered as a consequence of minimizing  $C_w$ , it was surmised that minimizing  $C_w = \int F\dot{l}$  resulted in solutions that had large phases with  $F = 0$  or  $\dot{l} = 0$  perhaps because both these phases had the cost identically zero. However, such superficial speculation seems insufficiently general, now that such optimal gait structure has been shown to be a property of minimizing a wider class of functions  $C_g$ , especially because the optimal structure is preserved even when the minimum of  $g(v)$  is not at  $v = 0$ . In appendix B, I prove using elementary geometric arguments, the optimality of this bang-coast-bang structure for a problem related to swinging a leg through a given amplitude. And indeed it is known that human leg swing during walking has muscle activation mostly at the beginning and the end of the swing [28]. It must be pointed out that a true bang-coast-bang structure is, in detail, at odds with common experience because it seems like humans are not exerting muscles maximally during slow-speed walking. Note also that while the similarity of optimal behaviour for  $C_g$  and  $C_w$  has been formally established in simple examples (§6 and appendix B), I only considered special cases of  $C_g$  for the locomotion optimization problems, which therefore does not constitute a complete proof that all  $C_g$  have similar optima. Finally, using the maximum available forces becomes non-optimal when there are force-derivative bounds—when using the maximum available force-derivative becomes optimal.

**5. ‘Compliant’ walking and running gaits.** Human gaits are relatively smooth and involve largely gradual growth of the various GRFs involved. Such smoothness has allowed these gaits to be approximated by ‘spring-mass models’, in which the leg is modelled as a massless linear spring, such as recently elaborated by a number of authors, notably [10] in the context of human walking [11,12]. The many ‘passive’ gaits obtained for these spring-mass models are all shown here to be work-optimal (and more generally  $C_g$ -optimal) in our springy actuated models.

On the other hand, when there are no springs in the biped’s leg, the optimal gaits here were mostly non-impulsive analogues of gaits with impulses. However, these impulsive gaits do become smoother when muscles are modelled more realistically. For instance, a muscle’s force cannot rise to some finite value instantaneously, but can only do so gradually, as allowed by its excitation–contraction dynamics. With such properties, the impulses get smoothed out, giving the gaits a compliant appearance.

**6. Linear spring-like leg behaviour.** Another feature, especially of human running gaits, is the roughly linear relationship between the leg force and leg compression. Interestingly, this roughly linear spring-like leg behaviour can be derived as a consequence of energy optimality in three vastly different situations. First, not surprisingly, the zero muscle work running gait of the running springy telescoping biped, in which the linear telescoping spring in series does all the leg work, has an exactly linear-spring-like leg behaviour in this limit (figure 16). More interestingly, the kned springy biped also showed a linear-spring-like

leg behaviour, but now only approximate because of the nonlinearity imposed by the knee kinematics (figure 19). At the other extreme, a springless telescoping biped or a springy biped with very high stiffness, now with a simple bound on the force rate or its derivatives, might have running gaits with a roughly linear spring-like leg behaviour—even in the absence of a spring (figure 10). Thus, a roughly linear spring-like leg behaviour is obtained in running whenever the leg force increases and decreases gradually and is unimodal—as in all these cases—independent of whether there is an actual spring involved.

**7. Walk-to-run transition.** One common theory of gait transition holds that animals switch to running because inverted pendulum walking requires high centripetal accelerations at high speeds, not achievable without tensional leg forces [2,30,64]. However, this theory is unsatisfactory as it does not explain why animals do not use other no-flight-phase gaits at higher speeds not requiring tensional forces, such as one with a horizontal centre of mass trajectory [30]. Here, as many others have also argued, it has been shown that energetics provide a logically satisfying explanation for gait transition.

For all the springless bipeds with the various cost functions and assumptions considered here, it was energetically optimal to transition from gaits with no flight phase at low speeds (walking) to gaits with a flight phase at higher speeds (running). This is a generalization of the analogous result in Srinivasan & Ruina [5]. In appendix C, I show why the gait transition happens at roughly the same speed-stride-length boundaries for a few different  $C_g$ -like cost functions.

The walk-to-run transition curves have not been explored well enough for different costs for the springy bipeds in this paper. But note that the springy bipeds can have work-free running at very slow speeds, if the leg forces are not constrained ([21]; also see fig. 5 of [12]). Therefore, a realistic walk-to-run transition is not predicted for work-minimizing springy bipeds for a range of not-too-high stiffnesses. But the walk–run transition claim does apply to springy bipeds with high enough spring stiffnesses and bounded leg forces, as they are similar to the springless bipeds. Also, Minetti & Alexander [23] obtained plausible gait transitions with a springy kneed model with  $C_{AM}$  and leg mass.

**8. Optimality of left–right symmetry.** Human and many other bipedal animal walking and running gaits have a left–right symmetry. That is, their left stance phases are at least approximately similar to their right stance phases. This symmetry is probably explained by energetic optimality given a bilaterally symmetric body.

First, all of the optimal gaits computed here for the springless bipeds for various costs had their left stance phases essentially identical to their right stance phases, despite allowing for asymmetry. In appendix E, assuming small step lengths, it is shown that a symmetric inverted pendulum walking gait always requires less work than an inverted pendulum gait with asymmetric steps. On the other hand, I did find that the cost landscape for  $C_w$  was relatively flat near the gait transition for the springless telescoping biped, so that

asymmetric gaits are close to but not quite optimal (suggesting that small perturbations of the model might make asymmetry optimal sometimes).

For springy bipeds with sufficiently compliant tendons, when muscle work  $C_w$  was minimized, it was found that both symmetric and asymmetric gaits (such as skipping; figure 14) were optimal. However, when the metabolic cost function is slightly modified with a cost for isometric force, it was often found that the flatness of the landscape was lost, the asymmetric gaits became non-optimal and the symmetric gait was picked out often as the optimal gait.

**9. Time-reversal symmetry in stance phases.** Many, but not all, of the optimal gaits discussed here had an additional symmetry property. Their stance phases had an approximate or an exact time-reversal symmetry. That is, these gaits look the same run forward or backward in time. Obviously, human gaits do not have this symmetry exactly, but they are not far away, especially if one looks at the GRFs and the centre of mass motion; it is this approximate symmetry that lets approximation of human running by the symmetric gaits of spring-mass models [9,10,12]. This approximate time-reversal symmetry in bipedal walking is therefore probably a consequence of energy optimality. Interestingly, note that this symmetry appears approximately true even when  $g(v)$  in  $C_g$  is far from an even function. Also, the optimal gaits for the springless models have another time-reversal symmetry, not demonstrated here: the gait that minimizes the cost  $C_g$  with  $g(v)$  is the time-reversed version of the gait that minimizes the cost  $C_g$  with  $g(-v)$ , assuming the force–velocity bound is not active, neglected or also reversed.

Finally, considering models with extended as opposed to point feet will modify these time-reversal symmetries somewhat.

**10. Horizontal-hip-trajectory walking.** It is sometimes argued [62] that horizontal-hip-trajectory walking (sometimes called ‘level walking’; [5]) might reduce work requirements from the legs because it was thought that muscle mechanical work was required mainly to lift and lower the centre of mass. Here, when I minimized the work costs, I never discovered this horizontal-hip-trajectory gait as an optimum—this horizontal-hip-trajectory gait requires more positive leg work than the optimal gaits, as also confirmed by experiments for metabolic costs [65,66]. See [30,67] and [5] for simple cost expressions ( $C_w$ ) for such level walking gaits; for constant speed level walking with double support,  $C_w = D^2/24H$ ; for non-constant speed level walking with no double support,  $C_w = D^2/32H$ . Comparing these cost expressions with  $C_w$  for inverted pendulum walking (approx.  $V^2D^2/32$ ), pendular running and impulsive running (approx.  $D^2/32V^2$ ) computed accurately using careful numerics show that level walking never minimizes  $C_w$  [5,30]. Such horizontal-hip-trajectory gaits are non-optimal for most other metabolic cost functions considered here as well.

**11. Walking on relatively straight legs.** Almost all the optimal gaits obtained here make use of essentially all the maximum leg length available to the

biped, at some part of the stance phase. For instance, in inverted pendulum walking, pendular running, impulsive running and their non-impulsive analogues, the leg is almost its maximum possible length for the whole of the stance phase. In the more compliant optimal gaits of the springy models, especially with a cost for isometric force, the leg length is maximal or close to maximal at the beginning and the end of stance phase. Classically, it has been argued that this tendency for straight legs, especially in human walking, is to reduce the muscle force requirements by taking the best advantage of the knee's kinematic singularity. However, it has been shown here (generalizing [5]) that even without a knee and even without a cost for force, it is useful to walk and run on straight legs for the springless models while minimizing a variety of metabolic costs. This trend is also clear from simple cost expressions for these idealized gaits.

#### 12. *Optimality of double support in walking.*

Minimizing  $C_w$  or  $C_g$  for springless bipeds at low speeds appears to produce walking gaits with no double support. That is, in these walking gaits, the push-off is entirely before the heel-strike with no overlap. This absence of double support appears to be true even when there is a not-too-low bound on the maximum leg force (say about 2–3). However, double support appears to become optimal when one has a bound on the *rate* of leg force (figure 9). This result shows that the optimality of push-off before heel-strike, advocated by some [3,4] is true for slow walking only when the leg forces can increase arbitrarily fast, as in a collisional impulse. Another reason to have a double support is if there is a strong cost on the leg force (say,  $C_{F2}$ ) without a knee singularity to reduce the muscle force—as for instance, is the case when the work is performed by the ankle joint. Finally, note that walking gaits with double support were also obtained as being optimal for the springy bipeds with substantial tendon compliance, while minimizing  $C_g$ -like costs.

**13. *Why might animals look like they minimize work instead of metabolic cost?*** Some animals have long compliant tendons in series with some of their muscles [40], so that during running, these tendons account for a large fraction of the positive and negative work of the musculo-tendon complex, while the muscle itself remains close to isometric [68,69]. This strategy is clearly muscle-work minimizing, at least for the muscles that are close to isometric. Even other animals appear to take advantage of the presence of tendons springs to reduce the amount of muscle mechanical work required [31,70–72]. Such running is literally close to the behavioural assumptions of the spring-mass model, although it is not clear if all major leg muscles are close to isometric (the proximal ones might not be). In any case, why care about ‘muscle work’ instead of metabolic cost? Here, it has been demonstrated that minimizing  $C_w$  is similar to minimizing a number of other costs. Thus, if animals appear to minimize work, it is probably because minimizing whatever is important to them (e.g. metabolic cost) ‘looks like’ minimizing work. Note again that minimizing work can be optimal even with substantial costs for isometric force.

**14. *Optimal robot gaits and biomimicry.*** While other researchers, especially roboticists [19,48,49,73] have determined the optimal gaits of models more complex than, but closely related to, the kneed springless model, it appears not to have been noted that a particular simplification of their models yields the classic inverted pendulum model and impulsive running as the optimum. (Note again that the  $M^2$  cost  $C_{F2}$  is a simple model of idealized motor electrical cost appropriate for robots). This similarity of idealized human gaits and idealized robot optimal gaits allows robots based on a blind mimicry of some aspects of human walking to be energetically economical even though the human metabolic cost function could be very different from a robot's electrical cost function. There is, of course, no *a priori* reason to expect that what is optimal for an animal would be optimal for a robot, but sometimes there might be coincidences such as those seen here. Note that I am not claiming that the corresponding optimal gaits are identical at any given  $[V, D]$ , but only that a similar set of gaits seem optimal overall.

Thus, one could perhaps conjecture that the under-actuated level-ground walking robots [59], inspired by the so-called passive dynamic robots, inherit their energy economy from their imitation of energy optimal gaits here, which have large energetically passive phases. Of course, in general, having many largely passive phases does not imply energy economy (even though energy economy has been equated with passivity in the literature); this gait structure just happens to be good for the many costs discussed here.

#### 15. *When are springs energetically useful?*

Whether or not a spring in series with a muscle can be energetically useful depends on the stiffness of the spring, the maximum muscle forces, the time scale of the stance phase and the properties of the metabolic cost function. Springs in series with muscles are useful for reducing the muscle work requirements during legged locomotion. But if the muscle forces are bounded, then the spring in series can store only as much energy as this maximum muscle force lets it. So a spring that is much too stiff can store little energy in this case; so a sufficiently stiff spring would be indistinguishable from having no spring at all. At the other extreme, a spring with very low stiffness in series with the muscle can also be energetically expensive. Imagine replacing your tendons with a very thin rubber band; clearly such low stiffness would make locomotion at normal speeds essentially infeasible and if feasible, expensive (see [30] for a related discussion about leg swing; also [32]). Thus, given a motion with some time scale, it is likely that there is a range of optimal tendon stiffnesses for reducing muscle work requirements [42].

Note that springs do not only help reduce work requirements. Springs are useful in the reduction of the more general metabolic cost  $C_g$  as well. Even more generally, it seems likely that springs help minimize metabolic cost when the metabolic cost function depends strongly on the muscle contraction speed. On the other hand, if the cost were a pure function of muscle force, for instance,  $C_{F2}$ , then clearly springs can have no direct effect on the cost—so the springy

bipeds are the same as the springless bipeds. This means that robots with electrical motors with perfect regeneration might be less amenable to cost reduction using springs in series with the motors. This is also a critique of the use of  $C_{F2}$  or other pure functions of muscle force in predictive biomechanics optimizations—all such optimizations should find that the presence of springs have only little or no direct effect in reducing the cost (springs may have a small indirect effect on the coordination through the muscle force–velocity relation).

#### 16. What dominates cost at an optimum?

First, consider a simple concrete example. Consider a function  $f$  that is a weighted sum of two smooth functions:  $f(x) = \lambda f_1(x) + (1 - \lambda)f_2(x)$ , where  $f_1(x) = (x - a)^2$  and  $f_2(x) = (x - b)^2$ . The optimal solution  $x^* = \lambda a + (1 - \lambda)b$ . By changing  $\lambda$  from 0 to 1, the optimum of  $f(x)$  shifts from  $x^* = b$ , the minimum of  $f_2$ , to  $x^* = a$ , the minimum of  $f_1$ . It can be shown that when  $\lambda \approx 1$ , even though  $x^*$  is determined by minimizing  $f_1$ , the cost  $f$  is dominated by the  $(1 - \lambda)f_2$  term. On the other hand, when  $\lambda \approx 0$ , even though  $x^*$  is determined by minimizing  $f_2$ , the cost  $f$  is dominated by the  $\lambda f_1$  term. This overall result is generically true for a weighted sum of any two smooth functions.

Thus, the part of the cost function that mostly determines the optimum might be only a small part of the optimal cost. In other words, just because something costs you a lot at the optimum does not mean that this aspect of the cost is the most important determinant of the optimum strategy. Rather, it might well be the less important determinant of the optimum. The more important determinant might have a much smaller fraction at the optimum.

Thus, for instance, it might be possible for the cost of locomotion to be largely explained by the cost of isometric force (if it is, as sometimes argued; [74,75]), while the optimal gait is largely explained by the minimization of work, say. This feature is observed in many of the zero muscle work  $C_g$ -optimal gaits of the springy bipeds here.

**17. Insensitivity to objective function.** As noted in the introduction, one goal of research on legged locomotion is to determine a hopefully simple function that when minimized under appropriate task constraints for a simple-enough biped model predicts the observed locomotor behaviour [26]. In this paper, I have shown that the optimal gait is insensitive to some objective function perturbations. See appendix D for one simple explanation for such insensitivity involving a non-smooth minimum. Insensitivity of optima to the objective functions means that it may be harder to infer a unique objective function purely from the observed behaviour alone (if we cannot measure the cost function as well). On the other hand, once the insensitivity is better understood (or even if not), the insensitivity helps by letting us make good predictions with a relatively poor model of the objective function.

#### 18. What about the predicted metabolic costs?

In the models here and in the following discussion, I do not consider the cost of swinging the legs, which is thought to be a small fraction (perhaps about one-

third) of the total cost [76]. In [5] and [30], I plotted certain aspects of the optimal work costs  $C_w$  for the springless biped models, and noted that the metabolic cost values estimated by  $C_w$  are within a factor of 2 of the metabolic cost estimated from  $\text{VO}_2$  experiments (e.g. figs 2.4 and 2.7 in [30] assuming  $b_1 = 4$  and  $b_2 = 0.8$ ). The leg work in these optimal gaits are also close to the so-called centre of mass work (sometimes called ‘external work’) computed from the animal’s GRFs. Without changing the optimal gaits at all, much better agreement with metabolic data could presumably be achieved by considering the more general cost  $C_g$  and choosing  $g(v)$  appropriately for a springless biped, whether kneed or telescoping. This would be somewhat unsatisfactory as it is known that springs do have a role to play in the energetics.

On the other hand, the springy biped models, while producing gaits that are compliant and having more realistic looking GRFs, suffer from too low an estimate of the metabolic cost, especially if the series spring is sufficiently compliant and is performing all the leg work. In this work-free limit,  $C_w = 0$  and  $C_g$  is mostly due to the integral of the muscle force. A slightly more realistic model with knees, tendon springs and leg mass owing to Minetti and Alexander also suffers from a similar underestimation of the metabolic cost while using  $C_{AM}$  (see fig. 4d in [23]). While this cost underestimation might potentially be fixed by manipulating  $g(v)$ , this manipulation might have to be substantially at odds with the data informing the  $C_{AM}$  cost. Also, the perfectly muscle-work-free optimal gait structure is probably unlikely to be a good approximation anyway, as it is thought that only about 50 per cent of the leg work is due to the springs in the legs and the rest is due to muscle work [53]. Overall, I suspect that a generalization of the metabolic cost model  $C_g$  to something that looks like, say,  $\int f(F)g(v) dt$  or  $\int h(F,v) dt$  might be indicated, or perhaps as others have suggested, a cost function that includes force derivatives.

**Summary.** In this paper, I have explored many objective functions, many biped models, under many situations. This exploration resulted in many qualitative results, which I suspect apply more generally than just for the situations considered and described here. I have only explored a small section of a very large number of possible simple models, and even within those simple models, only specific parameter combinations have been considered. As the space of possible bipeds is explored more carefully and more exhaustively, it is likely to be found that some of the claims and conclusions here are insufficiently general and must be revised and subsumed under more general results. A more careful and exhaustive analysis of each of the four biped models in this article is indicated and might provide a foundation for the construction and understanding of more complex and more realistic models. Also, in this article, I mostly derive the optimal gaits using numerical optimization and they are sometimes buttressed by analytical reasoning where appropriate. It is likely that many of these optimization calculations can be understood a bit better by attempting proofs of optimality, perhaps in simplified cases such as in [29].

In summary, while no single model examined here has all the features that we would like of a predictive model of steady legged locomotion, it has been shown here that a large number of qualitative features of human bipedal locomotion can be explained by the hypothesis of energy optimality, even when applied to only relatively simple biped models. It seems likely that a model not much more complex than the ones presented here could be much more successful.

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## APPENDIX A. MINIMIZING $F^2$ COSTS

Consider two actuators in parallel, exerting forces  $F_1$  and  $F_2$ , respectively. Say that it is required that the two forces sum to  $F_0$ . Then,  $F_1 = F_2 = F_0/2$  minimizes  $F_1^2 + F_2^2$ . Also, for a single actuator, if you want to produce an average force  $F$  while minimizing the integral of  $F^2$ , it is best to have a constant force rather an oscillatory force.

The above observation is implicated in the following: especially for small step lengths when the leg forces are close to vertical, the  $C_{F_2}$  optimal gait for the telescoping biped had a lot of double support; in fact, most of the stance phase tended to be double support. This is because a given force  $F$  over time  $T$ , produced by one leg roughly costs  $F^2 T$  and when produced by two legs costs  $2 \times (F/2)^2 T = F^2 T/2$ . For higher speeds and step lengths, a running-like gait with flight phases and single-stance phases appears optimal (figure 8).

## APPENDIX B. MINIMIZING $C_g$ FOR A SIMPLE TASK RESEMBLING SWINGING A LEG

Here I describe a simple task analogous to leg swing [30] for which it can be proved that minimizing  $C_g$  with a positive  $g(v)$  is identical to minimizing  $C_w$ .

Consider the mass  $m$  on a frictionless surface, whose horizontal position is denoted  $x$  (figure 20*a*). Say the task is to move the mass through a distance  $x_{\max}$  in time  $t_{\max}$ ; in effect, an average speed of  $v_{\text{avg}} = x_{\max}/t_{\max}$  is required. The mass must start at rest and end at rest ( $\dot{x} = 0$ ). There is one actuator, able to apply forces  $F(t)$  of any magnitude.

*What strategy minimizes  $C_w$  for this task?* I will show that the best strategy is to apply an impulse initially, so that the mass is immediately accelerated to a speed  $v_{\text{avg}} = x_{\max}/t_{\max}$ , let the mass coast the distance  $x_{\max}$  over time  $t_{\max}$ , and then decelerate to zero speed instantaneously at time  $t_{\max}$ . The positive work

required for this bang-coast-bang strategy is  $mv_{\text{avg}}^2/2$ . A short outline of a proof of optimality follows. This claimed optimal strategy uses a constant forward velocity equal to the required average velocity  $v_{\text{avg}}$  over essentially the whole time duration. An alternative strategy, with some finite duration spent at velocities less than  $v_{\text{avg}}$ , will necessarily also have a duration in which the mass travels faster than  $v_{\text{avg}}$ , to satisfy the average velocity constraint. This means that the mass has, at some point during its motion, an energy greater than  $mv_{\text{avg}}^2/2$ . This implies that the minimum required positive work ( $= \int Fv dt$  with  $F > 0$  and  $v > 0$ ) for such an alternative strategy must be greater than  $mv_{\text{avg}}^2/2$ , proving the optimality of the impulsive strategy, which only requires  $mv_{\text{avg}}^2/2$ . Figure 20*b* shows the path OABC taken by the optimal strategy in the  $x - \dot{x}$  phase plane. See [30] and [29] for similar arguments in similar contexts.

*What about minimizing  $C_g$ ?* It can be shown that exactly the same bang-coast-bang strategy, the path OABC of figure 20*b*, also minimizes  $C_g$  with a positive  $g(v)$  (where  $v = \dot{x}$ ) for this task. The proof is the same as above, except now ‘energy’ in the above proof is replaced with the quantity  $G(v) = \int_0^v g(v') dv'$ , which is monotonically increasing with  $v$  when  $g(v)$  is positive. Note that  $G(v)$  equals the cost  $C_g$  when  $F > 0$  and  $v > 0$ , and, therefore, has the same status that positive work has to  $C_w$ :  $C_g = \int Fg(v) dt = \int \dot{v}g(v) dt = \int g(v) dv$ . The constant mechanical energy contours in figure 20*b* now become constant  $G(v)$  contours, albeit with different spacing.

The similarity of the optimal strategies for the costs  $C_w$  and the more general  $C_g$  for this simple task foreshadows the similarity of the optimal walking and running strategies for the springless telescoping model. The sameness only requires that  $G(v)$  be strictly increasing, which is assured for positive  $g(v)$ . Convexity of  $g(v)$ , which was used in §6, was not required for the above proof. Also, in the above proofs, it was implicitly assumed that  $F$  could be positive or negative; it may be more physically plausible to have two muscles, one for acceleration (agonist) and another for deceleration (antagonist), but both the optimal strategy and the proof arguments would be essentially identical.

## APPENDIX C. A GAIT-TRANSITION THEORY FOR THE SPRINGLESS BIPEDS

In this section, I present simple expressions for the cost of the optimal gaits for the springless telescoping biped, minimizing a  $C_g$ -like cost—namely for inverted pendulum walking and impulsive running. Using these simple expressions, it is possible to determine when gait transitions should occur. The cost of transport is used to choose between different gaits; the cost of transport  $c_t$  is the non-dimensional quantity defined as the metabolic cost per unit weight and distance travelled. For simplicity, the arguments below about gait transitions assume that the gait choice is only between inverted pendulum walking and impulsive running gaits, and show that transition between just these

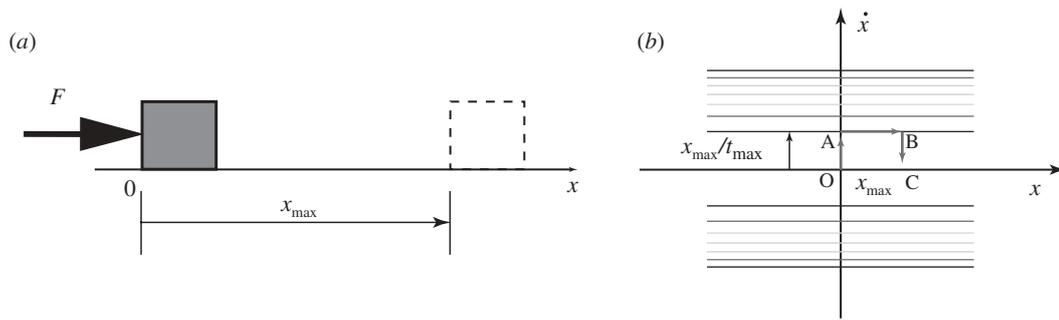


Figure 20. (a) The task is to move the mass from  $x = 0$  to  $x_{\max}$  in time  $t_{\max}$ , starting and ending at rest. (b) The optimal strategy is OABC, OA is instantaneous, AB is constant speed and BC is instantaneous. The series of horizontal lines are the energy contours. A non-optimal path will necessarily have to intersect a higher energy contour than AB, thus requiring greater positive work.

gaits should happen at roughly the same  $[V, D]$  boundaries for some different  $C_g$ 's for small step lengths.

**Impulses for inverted pendulum walking and impulsive running.** Especially for  $C_w$ , all the cost is incurred at the impulses: the heel-strike and push-off impulses for pendular walking and running, and the single pseudo-elastic impulse for impulsive running. The pendular phases of these gaits have no cost, as these phases involve no work, and in the case of the kneed biped, requires no knee moment. So for completeness, I recall the impulse magnitudes for gaits inverted pendulum walking and impulsive running; see [30] and [4] for details.

For inverted pendular walking (and pendular running), the push-off and the heel-strike impulses are each equal to  $v_c = v_- \tan \theta$ , where  $v_-$  is the velocity magnitude just before push-off,  $\theta$  is the leg angle with vertical and  $\tan \theta = D/4\sqrt{1 - D^2/16}$ . The speed  $v_-$  can be determined by numerical simulation or using approximations. A simple approximation, applicable for small  $D$  and not-too-small  $V$  is  $v_- \approx V$ , so that  $v_c \approx VD/4$ .

For impulsive running, the single vertical impulse can be broken up into a negative work impulse and a positive work impulse. The magnitude of each of these impulses is equal to the vertical velocity just before impact:  $v_c = T/4 = D/4V$ . There are no approximations in this expression. Note again that in this paper,  $D$  stands for stride length, whereas in Srinivasan & Ruina [5] and Srinivasan [30],  $D$  stands for step length.

**Work cost  $C_w$ .** For both the springless telescoping biped and the kneed biped, it can be shown that the cost of transport  $c_t$  for a positive work cost ( $C_w$  with  $b_1 = 1$  and  $b_2 = 0$ ) is given by:

$$c_t = \frac{C_w}{D} = \frac{v_c^2}{2D}. \quad (\text{C } 1)$$

where  $v_c$  is the non-dimensional magnitude of the push-off impulse [4,30]. Substituting the values for  $v_c$  derived in the previous paragraphs,  $c_t \approx V^2 D/32$  for inverted pendulum walking. For impulsive running,  $c_t = D/32V^2$ . If the choice were restricted to just between these two gaits, a comparison of these costs shows that walking is better for  $V < 1$  and running for  $V > 1$ . This is a small- $D$  approximation. The exact phase boundaries are in Srinivasan [30] and Srinivasan &

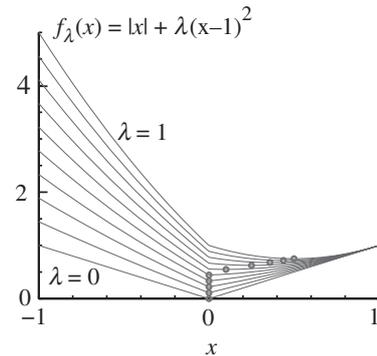


Figure 21. Robustness of a non-smooth minimum. The function  $f_\lambda(x) = |x| + \lambda(x-1)^2$  is shown for a sequence of  $\lambda$  from 0 to 1. The minima of  $f_\lambda(x)$  (shown as little circles) remains exactly at  $x = 0$  for  $\lambda = 0$  to 0.5, and is greater than zero for  $\lambda > 0.5$ .

Ruina [5]. The key observation here is that the cost expression (equation (C 1)) is a monotonic function of  $v_c$  for a given  $D$ , and therefore, the gait transitions happen when the  $v_c$  for one gait becomes larger than another. Such observations apply to the springless kneed biped as well.

**Convex cost  $C_g$ .** The  $C_g$  cost of the positive work impulse is given by  $\int_0^{v_c} Fg(v) dt = \int_0^{v_c} \dot{v}g(v) dt = \int_0^{v_c} g(v) dv = G(v_c) - G(0)$ . The negative work impulse has a cost equal to  $G(0) - G(-v_c)$ . The cost of the pendular phase is equal to  $g(0)t_{\text{pendular}}$ , where  $t_{\text{pendular}}$  is the time spent in the pendular phase. The cost of transport will then be  $c_t = [G(v_c) - G(-v_c) + g(0)t_{\text{pendular}}]/D$ . If  $g(0)$  were zero, the cost  $c_t$  would be a monotonic function of  $v_c$  for a given  $D$  and, therefore, the gait transition boundaries will be exactly the same as in the earlier two cases. When  $g(0)$  is not equal to zero, the gait transition boundaries are likely to be a little different from the earlier cases. Note that  $g(0)$  for  $C_{AM}$  is relatively close to zero (figure 2c).

#### APPENDIX D. INSENSITIVITY OF WORK MINIMA TO AN ADDITIVE $C_{F2}$ COST

Consider the non-smooth function  $f_\lambda(x) = |x| + \lambda(x-1)^2$ . For  $\lambda = 0$ ,  $f(x) = |x|$  and is minimized at  $x = 0$ . Figure 21

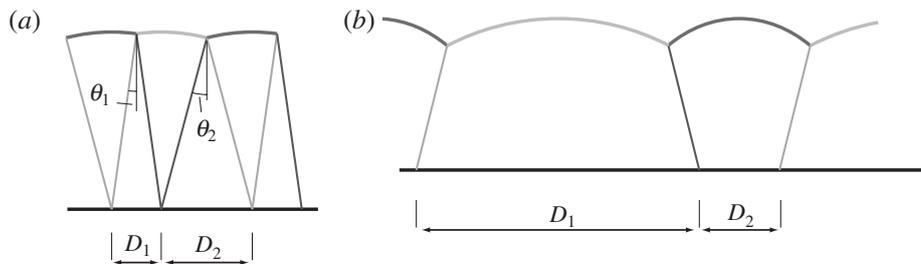


Figure 22. Idealized asymmetric gaits. (a) Asymmetric inverted pendulum walking, with two unequal step lengths in a stride. The leg length is constant during most of the stance phase, equal to the maximum possible leg length, except for infinitesimal extensions and contractions during push-off and heel-strike impulses. (b) Asymmetric impulsive running gait, with two unequal steps in a stride.

shows  $f_\lambda$  for a sequence of  $\lambda$ . The minimum remains exactly at  $x=0$  for a range of  $\lambda$  around  $\lambda=0$  (a 'dead zone') and then changes gradually from  $x=0$ .

It can be shown that dead-zone behaviour for minimum around  $\lambda=0$  is *generic* for functions of the form  $f_\lambda(x) = f_1(x) + \lambda f_2(x)$  ( $x \in \mathbb{R}^n$ ) when the  $f_1(x)$  has a 'non-smooth minimum' at  $x=a$  such that the directional derivative of  $f_1(x)$  along every feasible direction is positive (bounded away from zero). The proof is elementary: if the directional derivatives of  $f_2(x)$  at  $x=a$  are bounded, the directional derivatives of  $f_\lambda(x)$  at  $x=a$  remain positive for a range of  $\lambda$  around  $x=0$ , thus making  $x=0$  remain a minimum for this range. The set of feasible directions is assumed independent of  $\lambda$ .

Arguments similar to these can presumably be used to show the robustness of optimality of the zero work strategies to small cost function perturbations of the form  $\lambda \int F^2 dt$  for the springy bipeds. For a given speed, stride length and spring stiffness, there is only one (locally isolated) symmetric work-free gait. The work cost for this gait probably satisfies the conditions above: that any small perturbation to this gait, satisfying the speed and stride length constraints, will result in a small work cost whose size is the same order as the perturbation to the gait (say measured by changes in leg forces), thus showing the positivity of the directional derivatives in every direction, as required by the above proof of insensitivity.

## APPENDIX E. OPTIMALITY OF SYMMETRIC WALKING

Consider the asymmetric walking gait shown in figure 22a, in which the two steps that make up a stride are unequal. The two push-offs in a stride are accomplished with different leg orientations, namely with leg angles  $\theta_1$  and  $\theta_2$  as shown in the figure. For small leg angles, it can be shown that the positive work performed during the push-off scales like  $V^2 \theta^2 / 2$  (see appendix C and [30]). Given  $\theta_i \approx D_i / 2$ , the total cost (positive work) over the two steps (one stride) would be  $V^2(D_1^2 + D_2^2) / 8$ . Given  $D_1 + D_2 = D$ , minimizing the cost for a given  $V$  and  $D$  is like minimizing  $D_1^2 + D_2^2$ , accomplished by setting  $D_1 = D_2 = D/2$ . The reasoning is identical to that in appendix A. Thus, symmetric inverted pendulum walking is best among all inverted pendulum walking gaits. This proof assumes small leg angles, but the claim is generally true.

Similarly, it can be shown that symmetric impulsive running, with equally spaced vertical impulses, is better than the asymmetric impulsive running gait in figure 22b. Because the two costs (for symmetric and asymmetric impulsive running gaits) are the same up to leading order in  $D$  [4], the proof of optimality would be somewhat longer than the above proof for walking. But see [30] for a demonstration of the claim that a symmetric impulsive running gait is better than an asymmetric running gait in which the two impulses are close together, analogous to skipping.

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# Supplementary information for: Fifteen observations on the structure of energy-minimizing bipedal gaits in many simple biped models

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## 1 Numerical solution of the gait optimization problems

**Overview.** The central “gait optimization problem” in this paper is to obtain the periodic gait with a specified average horizontal speed  $V$  and specified stride length  $D$  that minimizes the metabolic cost  $C$  over the gait period, subject to other constraints. The various biped models considered here are all capable of an infinite variety of gaits. In fact, given the same leg force constraints (and other constraints as appropriate), the four bipeds are capable of the same set of gait trajectories. This set of possible gaits is an infinite dimensional space. Any particular gait is completely described by specifying the two leg forces as functions of time  $F_{1,2}(t)$  (or equivalently the time derivative of the leg forces with their starting values), the initial state of the biped (position and velocity at, say, the beginning of the stride), the duration, timing, and sequences of the single stance, double stance, and flight phases, and the horizontal foot positions during the stance phases.

The infinite-dimensional optimization problem is made tractable by considering only a finite-dimensional gait space, obtained by describing the leg forces using finitely many parameters, reducing the “trajectory optimal control problem” [6] to a “nonlinear programming problem” [3, 4]. The computational methods used here for the solution of the gait optimization problem are closely related to those described in [5, 4], but are modified as described below to handle the more general and sometimes more numerically ill-conditioned calculations attempted here.

**Definition: the “primary structure” of a gait.** It is convenient to sub-divide the space of all possible bipedal gaits based on the sequence of single stance phases, flight phases, and double stance phases that characterizes a gait. I introduce a new notation here. A simple sequence of letters is used to characterize each gait structure, where  $f$  denotes a flight phase,  $s_i$  denotes the single stance phase of the  $i$ -th leg, and  $d$  denotes a double stance phase. For instance, a human running gait can be denoted by the sequence  $[s_1 f s_2 f]$ ; that is, a single stance phase, followed by a flight phase, another single stance phase, and again a flight phase. A human walking gait with double stance phases can be denoted by the sequence  $[s_1 d s_2 d]$ . Any cyclic permutation of a sequence also represents the same gait. Thus, for instance, human running can also be described by the sequences  $[f s_2 f s_1]$ ,  $[s_2 f s_1 f]$ , and so on. I call such sequences describing the gait as the “**primary structure of the gait**,” in analogy to the term’s usage in describing protein structure. One can equivalently use the sequence of footfalls and foot-liftoffs to specify such primary gait structures.

**Discrete and continuous variables.** The primary structure of a gait is a discrete-valued and not a continuous-valued variable, while all other variables describing a gait — such as forces, initial conditions, and durations of stance and flight phases — are continuous-valued. Because it is difficult to optimize over a space which has a mixture of discrete and continuous valued variables (such problems called “mixed integer programming” problems in the optimization literature), the optimization problem here is solved in two stages. For each  $V$  and  $D$  considered, I loop over many of the possible primary structures. For each primary structure, I find the optimum gait from among gaits having that primary structure. Having found the optimal gait for each primary structure, the gait with the least cost is picked from these finitely many optimal gaits. Ideally, one should loop over all possible primary structures.

**Relevant primary structures for a biped.** In this paper, I only consider gaits in which each leg has at most one stance phase over a single stride period. This is quite a general class of gaits. For the purposes of optimization, only three different gait structures having this property are considered, namely  $[s_1 f s_2 f]$  (e.g., running),  $[s_1 d s_2 d]$  (e.g., walking with double stance) and  $[f s_1 d s_2]$  (e.g., unilateral skipping, also called bipedal galloping, with double stance). Many other sequences with at most one stance phase per leg per stride can be obtained as special cases of these three. For instance, a gait with two single stances  $[s_1 s_2]$ , but with no double stances or flight phases can be obtained as a limit of  $[s_1 f s_2 f]$  with vanishing flight phases, as a limit of  $[s_1 d s_2 d]$  with two vanishing double stances, or as a limit of  $[f s_1 d s_2]$  with vanishing flight and double stance phases. Note however, that there are not just three, but six possible primary gait structures for a biped in which each leg has at most one stance phase during a gait cycle. The other three not considered here are  $[s_1 d s_1 f]$ ,  $[s_2 d s_1 f]$ , and  $[s_2 d s_2 f]$ . If the two legs are assumed interchangeably symmetric, two of these three primary structures not considered are the same as two considered here. The combinatorics of the primary gait structures of an  $N$ -legged animal will be considered in some detail in a separate article.

## 1.1 Spring-less telescoping biped

In this section, I describe the numerical methods in the context of the spring-less telescoping biped in detail. The optimization for the other bipeds or other conditions require some minor modifications, described in brief in the subsequent sections.

**Continuous variables defining the gait.** First assume one of the primary gait structures, say,  $[s_1 f s_2 f]$ . Any gait with this primary structure is then parameterized with the following variables: the time duration of the two stance phases  $t_{s1}$  and  $t_{s2}$ , the period of the two flight phases  $t_{f1}$  and  $t_{f2}$ , the initial conditions for the two stance phases  $(x_{10}, \dot{x}_{10}, y_{10}, \dot{y}_{10})$  and  $(x_{20}, \dot{x}_{20}, y_{20}, \dot{y}_{20})$ , the leg forces over the two stance phases  $F_1(t)$  and  $F_2(t)$  respectively, and the ground contact positions  $x_{c1}$  and  $x_{c2}$  of the two feet during the two stance phases. The leg force functions are parameterized as being piecewise linear and continuous, defined on an evenly spaced time-grid over each stance phase- $i$ , where  $N$  is the number of grid-points for each stance phase. The force values at the grid points defining the piecewise-linear force functions are denoted  $F_{1j}$  and  $F_{2j}$ , where  $j = 1, 2, \dots, N$ . Thus, a gait is defined by specifying  $\mathcal{Z} = (t_{s1}, t_{s2}, t_{f1}, t_{f2}, x_{10}, \dot{x}_{10}, y_{10}, \dot{y}_{10}, x_{20}, \dot{x}_{20}, y_{20}, \dot{y}_{20}, F_{1j}, F_{2j}, x_{c1}, x_{c2})$ , a total of  $2N + 14$  variables. This is just a convenient representation of the gait, one that enables the simulation of the first stance phase independently of the second.

**Evaluating dynamics and metabolic cost.** For any given value of  $\mathcal{Z}$ , the equations of motion given in the main article can be integrated to obtain the center of mass position  $(x(t), y(t))$  over the two stance phases. The center of mass motion  $(x(t), y(t))$  over the flight phases are obtained by integrating the flight equations starting from the final state of the corresponding stance phases. The “objective function” which is the metabolic cost is also evaluated during the integrations over the stance phases, by adding a differential equation for the cost  $C$  (i.e.,  $\dot{C} = \dots$ ). To maintain high accuracy, integration of the differential equation is always done from grid-point to grid-point. This is because the right hand side of the differential equation is non-smooth at the grid points (because of piecewise linear forces). Most smooth ODE integration methods assume the existence of higher derivatives. They would commit large errors and would likely be unacceptably noisy (that is, inconsistent [1]) if integrated over a non-smoothness. I use MATLAB’s variable step-size integration function `ode45` with a relative error tolerance of about  $10^{-10}$ , so that the derivative estimates are good. I chose a variable step-size integrator, as opposed to a noise-free constant step-size integrator [1], so that the integration is reasonably stable in the presence of some numerical stiffness or near-non-smoothness. Care is taken to make the optimization problem, namely the objective function and the constraints, sufficiently smooth while remaining close to the original optimization problem. Some smoothing techniques are outlined later in this article. Smoothness is critical to reliable convergence of the optimization iterations, because the optimization algorithms used assume continuity of second derivatives. When there is true or near- non-smoothness, the optimizer might either fail to converge (and know that it has failed) or halt at a non-optimum, fooled by poor derivative estimates.

**Linear and nonlinear constraints.**  $\mathcal{Z}$  cannot assume arbitrary values, but is restricted by various equality and inequality constraints, some linear and some nonlinear in  $\mathcal{Z}$ , as listed below. The nonlinear constraint violations are evaluated after the integrations of the equations of motion.

- A leg length inequality constraint  $l_i \leq 1$  is enforced at finitely many grid points within the respective stance phases. The grid used to enforce such path constraints is usually taken to be denser than the control grid  $N$ , usually adding one or more grid points in between the control grid points.
- The periodicity constraint requires that the state at the end of the second flight phase is equal to the state at the beginning of the first stance phase except for an horizontal translation of  $D$  in  $x$ :  $x(t_{s1}+t_{f1}+t_{s2}+t_{f2}) = x_{10} + D$ ,  $\dot{x}(t_{s1} + t_{f1} + t_{s2} + t_{f2}) = \dot{x}_{10}$ ,  $y(t_{s1} + t_{f1} + t_{s2} + t_{f2}) = y_{10}$ , and  $\dot{y}(t_{s1} + t_{f1} + t_{s2} + t_{f2}) = \dot{y}_{10}$ .
- The continuity of state requires that the state at the end of the first flight phase is equal to the state at the beginning of the second stance phase:  $x(t_{s1} + t_{f1}) = x_{20}$ ,  $\dot{x}(t_{s1} + t_{f1}) = \dot{x}_{20}$ ,  $y(t_{s1} + t_{f1}) = y_{20}$ , and  $\dot{y}(t_{s1} + t_{f1}) = \dot{y}_{20}$ .
- The stance and flight phases must sum to the total stride period,  $t_{s1} + t_{f1} + t_{s2} + t_{f2} = T$ , a linear constraint.
- The forces are bounded:  $F_{min} \leq F_i \leq F_{max}$ , when only simple force bounds exist. When there is a force-velocity relation, I enforce it as an inequality constraint at grid-points over the stance phase:  $F \leq F_0\varphi(\dot{l})$ .
- The stance phases satisfy  $0 < \epsilon \leq t_{s1,s2} \leq T$  and the flight phases satisfy  $0 \leq t_{f1,f2} \leq T$ .
- The initial states are constrained to be within a sufficiently large box. For instance,  $-2d_{stride} \leq x_{10} \leq 2d_{stride}$ ,  $0 \leq y_{10} \leq 3$ , etc. These reasonable inequality constraints on the optimization variables do not materially affect the optimization result as they are never active at the optimal solution. These help rule out gaits that start underneath the ground, for instance (note that there is no explicit modeling of the ground).

**Optimization methods and software.** All calculations were performed within MATLAB. As noted, the differential equations were integrated using `ode45`. All the optimization calculations were performed using MATLAB’s optimization solver `fmincon`, a gradient and Hessian based algorithm for solving nonlinearly constrained optimization problems. This solver requires the definition of two functions – one that evaluates the metabolic cost given  $\mathcal{Z}$  and another that evaluates the constraint violations given  $\mathcal{Z}$ . These two functions are constructed as described in the previous two paragraphs.

When some of the optimizations were repeated with SNOPT [2], a different robust implementation of Sequential Quadratic Programming, the converged optimal solutions were essentially the same as those obtained with `fmincon`.

For most of the optimization results displayed in the paper, I used roughly the same grid resolutions of about  $N = 10$  to  $N = 12$ . Convergence of the optimization to the same solution when started from multiple initial seeds suggest but not prove the globality of the obtained optimal solutions. Also, in the rare event that the optimization converges or stops at multiple solutions, I pick the solution with the least cost. Ideally, one should try many very different initial seeds for the optimization and observe whether all of them converge to the same optimal solution. I usually considered only one or two initial seeds for each optimization, but one that was sufficiently far away from the putative optimum. Nevertheless, I suspect globality of most of the optima here (for the given gait structure and up to incomplete convergence). Convergence of the optimization iterations is not perfect, but quite close, as can be seen in the smallness of the deviations from left-right symmetry in the claimed minima (assuming symmetry is optimal); in such cases, while it is found that the same optimization repeated with an explicit symmetry constraint reduces the cost slightly, such reduction is very small (fraction of a percent). Another indication of the overall quality of the optimization is the convergence to provably optimal solutions, for instance, the zero muscle work solution while minimizing  $C_w$  for the springy bipeds.

**Other gait structures.** Note again that the optimization described above is restricted to gaits having a primary structure  $[s_1 f s_2 f]$ . The details of the optimizations with other primary structures are quite analogous to those described above, with only minor changes to accommodate the changes in the gait structure. The major constraints, as before, are the leg length constraints, periodicity over a period, continuity of state across different phases, and the force bounds (or appropriate force-velocity constraints). In the cases where there is a single stance and a double stance phase, each such gait phase is described by  $N$  grid points and each such phase has its own initial conditions in  $\mathcal{Z}$ , with appropriate continuity conditions added.

## 1.2 Spring-less kneed biped

The gaits are defined with the same set of variables  $\mathcal{Z}$  as before. The equations of motion are integrated exactly as before. The only difference in the various function evaluations is due to the knee’s kinematic singularity. Because the leg force and time of stance are used to define the gait, if the leg forces are not chosen carefully, the leg length would exceed the maximum (equal to unity) during the stance phase. When the leg length exceeds unity, the knee angle and knee angular velocity are not defined; for instance,  $\alpha = \cos^{-1}(l)$  is complex-valued for  $l > 1$ . A related problem occurs when one needs to divide by a quantity such as  $\sin(\alpha)$  which goes to zero when  $l = 1$ . We solve both these problems by replacing every occurrence of the inverse cosine by an approximation to the inverse cosine, that returns real-values for all  $l$ , can be made arbitrarily close to the inverse cosine when  $0 \leq l \leq 1$ , and is never zero, as described in section 2.3 of this supplementary document. This replacement of the inverse cosine does not affect the results of the optimization materially because the leg length constraint  $l \leq 1$  is satisfied at the optimum.

## 1.3 Springy telescoping biped

Here, instead of approximating the leg forces  $F(t)$  as being piecewise linear, I approximate the leg force rate  $\dot{F}(t)$  with a piecewise linear function. This is because the leg force rate plays a relatively central role in the metabolic cost function and therefore, it is useful for the leg force to be a continuous function (which would not have been the case had the leg force had been piecewise linear). The leg forces are assumed to start and end at zero. The leg forces are obtained by integrating the leg force rates, starting from zero. As noted earlier, the spring force equals the leg force  $F = -k(l_{spring} - l_0)$ , the spring length rate is  $\dot{l}_{spring} = -\dot{F}/k$  and the telescoping actuator length rate is  $\dot{l}_{actuator} = \dot{l} - \dot{l}_{spring}$ . The leg force is constrained to begin and end at zero during each stance phase, so as to avoid a stance phase beginning or ending with already stored elastic energy, not directly accounted for in the stance energetics.

## 1.4 Springy kneed biped

Just as for the springy telescoping biped, I use piecewise linear leg force rates here for the description of the leg forces and the inverse cosine regularization described in section 2.3 of this supplementary document.

## 1.5 Note on gait structures considered

While ideally all six primary gait structures should be considered at every  $[V, D]$ , I most often considered the two gait structures  $[s_1 f s_2 f]$  and  $[s_1 d s_2 d]$ . The third gait structure  $[f s_1 d s_2]$  was also examined, but only at very few  $[V, D]$  combinations and conditions described in detail in this paper. In this sense, there might potentially be skipping-like gaits that are better than walking- and running-like gaits in parameter regimes not considered, as also noted in the discussion of the main text of the paper. The other three possible gait structures, as noted earlier, were never considered here, but should be in a more comprehensive future work.

# 2 Some smoothings and regularizations

Sequential Quadratic Programming, our optimization method of choice, assumes that the functions and the constraints have bounded second derivatives [3]. The following tricks help us enforce such smoothness on our optimization problems.

## 2.1 Positive part and the absolute value functions

A smooth approximation of the absolute value function is  $|x| \approx \sqrt{x^2 + \epsilon^2}$  where  $\epsilon$  is small. Analogously, the positive part function  $[x]^+$  is approximated as  $(x + \sqrt{x^2 + \epsilon^2})/2$  and the negative part function  $[x]^-$  is approximated as  $(-x + \sqrt{x^2 + \epsilon^2})/2$ . We use these in evaluating the work cost  $C_w$ , with  $\epsilon = 0.01$ . See [4] for details of convergence in  $\epsilon$  and for examples of other such smooth approximations.

## 2.2 Smoothing a piecewise-defined function

Often empirical data such as the force-velocity relation or the empirical metabolic energy consumption is best approximated by a piecewise defined function. Discontinuities in the function values or derivatives at the junction of the pieces may present problems either during ODE integration or during optimization. Here I suggest a smooth approximation to such potentially problematic piecewise defined functions.

Consider the function

$$f(x) = \begin{cases} f_p(x) & \text{if } x \geq 0 \\ f_n(x) & \text{if } x < 0 \end{cases} \quad (1)$$

whose pieces  $f_n$  and  $f_p$  are themselves as smooth as we require, but the function  $f$  as a whole may not be. Then, we may define a smooth approximation  $\bar{f}(x)$  as follows.

$$\bar{f}(x) = \begin{cases} f_p(x) & \text{if } x \geq \epsilon \\ g(x) & \text{if } -\epsilon < x < \epsilon \\ f_n(x) & \text{if } x < -\epsilon \end{cases} \quad (2)$$

where  $g(x)$  is the unique polynomial of degree  $2q-1$ , such that the function  $\bar{f}(x)$  is continuous and has continuous derivatives at  $-\epsilon$  and  $\epsilon$  up to derivative order  $q-1$ . When  $\epsilon \rightarrow 0$ ,  $\bar{f}(x)$  converges pointwise to  $f(x)$ . The function  $\bar{f}$  can be made arbitrarily differentiable by picking  $q$  appropriately.

## 2.3 Regularizing inverse cosine, etc.

The inverse cosine function  $\cos^{-1}(r)$  is not real-valued for  $r > 1$ . Sometimes during the course of optimizations for kneed bipeds, we might wish to perform this function evaluation for  $r > 1$ ; for instance, to compute the knee angle  $\alpha = 2\cos^{-1}(l)$  from the leg length  $l$ . Because the angle  $\alpha$  always occurs inside trigonometric functions, it is sufficient to find real-valued and smooth approximations of  $\sin(\alpha/2)$  and  $\cos(\alpha/2)$  in terms of  $l$ . First, by definition,  $\cos(\alpha/2) = l$ , which is smooth and real valued. Second,  $\sin(\alpha/2) = \sqrt{1-l^2}$ . But  $\sqrt{1-l^2}$  can be complex-valued and its absolute value has infinite first derivative at  $l = 1$ . Also, we would like the approximation to  $\sin(\alpha/2)$  to not go to zero at  $l = 1$ , because we may divide by  $\sin(\alpha/2)$  in some expressions. Satisfying these requirements is the following approximation of  $\sin(\alpha/2)$ :

$$l_s = \frac{(l-1) - \sqrt{(l-1)^2 + \epsilon}}{2} + 1 \quad (3)$$

$$\sin(\alpha/2) \approx \sqrt{1-l_s^2} \quad (4)$$

where  $l_s$  is an approximation to  $l$  when  $l \leq 1$  and always greater than zero when  $l > 1$  (similar to those in Appendix 2.1). The use of this approximation effectively mollifies the kinematic singularity at the knee; taking  $\epsilon \rightarrow 0$  makes the approximation more and more accurate, eventually restoring the singularity at  $l = 1$ .

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